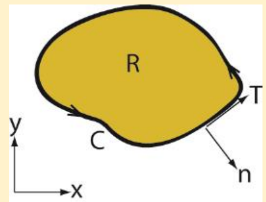
Green's Theorem

In a Nut Shell: Green's theorem gives the relationship between a line integral around a simple closed curve, C, and a double integral over the plane region R bounded by *C*. It is a special two-dimensional case of the more general Stokes' theorem.

Green's theorem expressed in its **standard form** is

$$\int_C P\,dx + Q\,dy = \int\int_R \left[\partial Q/\partial x - \partial P/\partial y \right] dA$$

where C is a curve enclosing the region, R, with element of area dA. The curve, C, is said to be positively oriented when traveling counterclockwise around C keeping the region, R, enclosed to the left.



Note: The partial derivatives must be continuous throughout R else you will need to modify the region to avoid discontinuities such as in a region R that is not simply connected. i.e. A region that contains a hole.

Green's theorem can also be expressed in its "**curl form**".

$\mathbf{F} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$

So $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_C P\,dx + Q\,dy$$

where **T** is the unit tangential vector to the curve, C, **n** is the unit normal vector to the curve and ds is the arc length along the curve. See the figure above.

Now $$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = [\partial Q/\partial x - \partial P/\partial y]\,\mathbf{k}$$

So $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int\int_R \text{curl}_z\,\mathbf{F}\;dA$$ (curl form of Green's Theorem)

where $\text{curl}_z\,\mathbf{F}$ is the z-component of curl $\mathbf{F} = \text{curl}\,\mathbf{F} \cdot \mathbf{k}$

Green’s theorem can also be expressed in its “**divergence form**”.

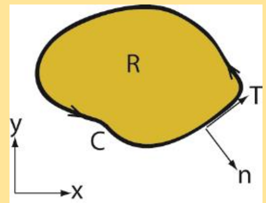
Let **n** be the unit outward normal to the curve, C.
Here $\mathbf{n} = dx\,\mathbf{i} - dy\,\mathbf{j}$

and $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$

So $\mathbf{F} \cdot \mathbf{n} = P\,dx - Q\,dy$

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C P\,dx - Q\,dy = \int\int_R [\partial Q/\partial x + \partial P/\partial y]$$

Finally $$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int\int_R \text{div}\,\mathbf{F}\, dA$$



Summary:

Green's Theorem in **Standard Form**:	$\int_C P\,dx + Q\,dy = \int\int_R [\partial Q/\partial x - \partial P/\partial y]\, dA$
Green's Theorem in **Curl Form**: (vector form)	$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_R \text{curl}_z\,\mathbf{F}\, dA$ $\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_R (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\, dA$
Green's Theorem in **Divergence Form**: (vector form)	$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int\int_R \text{div}\,\mathbf{F}\, dA$

Example: Use Green's Theorem to Evaluate $I = \int_C y^2\,dx + xy\,dy$

around the closed curve, C, bounding the region, R, where R is the ellipse

defined by $(x/3)^2 + (y/2)^2 = 1$.

Strategy: Apply the standard form of Green's Theorem to evaluate the line integral

$$I = \int_C P\,dx + Q\,dy = \iint_R [\partial Q/\partial x - \partial P/\partial y]\,dA$$

Here $P(x,y) = y^2$, $Q(x,y) = xy$ and the closed curve, C, is the

Now $\partial Q/\partial x = y$ and $\partial P/\partial y = 2y$ so $\partial Q/\partial x - \partial P/\partial y = -y$

Thus the area integral becomes $\iint_R -y\,dA = -2\int_{y=-2}^{y=2}\int_{x=0}^{x=3\sqrt{[1-(y/2)^2]}} y\,dx\,dy$

$$I = -2\int_{y=-2}^{y=2} y\; x\Big|_{x=0}^{x=3\sqrt{[1-(y/2)^2]}}\,dy = -6\int_{y=-2}^{y=2} y\sqrt{[1-(y/2)^2]}\,dy$$

let $u = 1-(y/2)^2$, $du = -(1/2)\,y\,dy$, $y\,dy = -2\,du$

$$I = 12\int \sqrt{u}\,du = 8\,[1-(y/2)^2]^{3/2}\Big|_{-2}^{2} = 0 \qquad \text{(result)}$$

Note this result makes sense since $\iint_R -y\,dA$ should equal zero when encompassing the region, R, by going completing around the ellipse (the areas on both sides of the y-axis cancel out).

Example: Evaluate the line integral: $\int_C (7y - e^{\sin x})dx + (15x - \sin(y^3 + 8y)\,dy$

where C is a circle of radius 3 centered at (5, -7). Note that this is a very difficult

integral to evaluate. So try simplifying the calculation using the RHS of Green's Theorem.

$\partial Q/\partial x = 15$, $\partial P/\partial y = 7$; $\iint_R[\partial Q/\partial x - \partial P/\partial y]\,dA = \iint_R 8\,dA = 8\,(\pi\, 3^2) = 72\pi$ (result)

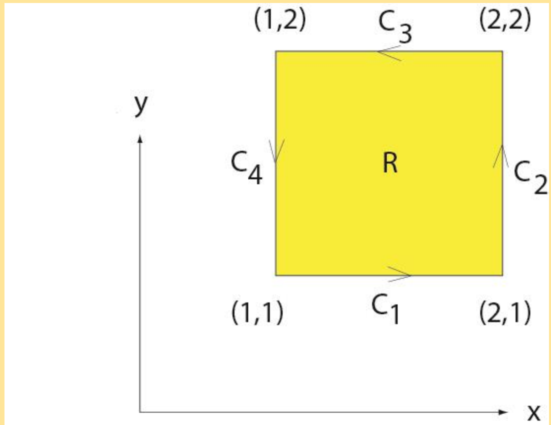
Note the simplification that results in using Green's Theorem.

Example: Apply Green's Theorem to evaluate the work done by the force vector, **F**, going

completely around the square region, R, (shown below) counterclockwise.

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R [\partial Q/\partial x - \partial P/\partial y]\,dA$$

where $\mathbf{F} = [\,y/(x^2+y^2)\,]\,\mathbf{i} - [\,x/(x^2+y^2)\,]\,\mathbf{j} = P\,\mathbf{i} + Q\,\mathbf{j}$



Method 1: Apply $\oint_C = \iint_R [\partial Q/\partial x - \partial P/\partial y]\, dA$

Now $\partial Q/\partial x = (x^2 - y^2) / (x^2 + y^2)^2 = \partial P/\partial y$

Note that these partial derivatives exist throughout the region, R. Thus the double integral over R equals zero since $\partial Q/\partial x = \partial P/\partial y$.

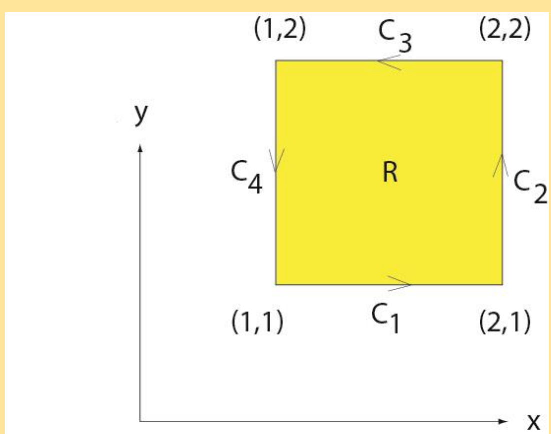
Method 2: Calculate the work done by evaluating the line integral directly around the curve, C, encompassing the region R.

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R [\partial Q/\partial x - \partial P/\partial y]\, dA$$

Strategy: In calculating the line integral on the left hand side of Green's Theorem

$\int_C \mathbf{F} \cdot d\mathbf{r}$ **Strategy:** Break the path C into four parts,

C_1, C_2, C_3, and C_4 proceeding counterclockwise starting at (1,1).



i.e. C_1 goes from (1,1) to (2,1), C_2 is from (2,1) to (2,2), C_3 is from (2,2) to (1,2), and finally C_4 is from (1,2) returning to (1,1).

$$\mathbf{F}\cdot d\mathbf{r} = [(y\,\mathbf{i} - x\,\mathbf{j})/(x^2+y^2)]\cdot[dx\,\mathbf{i} + dy\,\mathbf{j}] = (y\,dx - x\,dy)/(x^2+y^2)$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r} + \int_{C_4} \mathbf{F}\cdot d\mathbf{r}$$

Note: On C_1 $y = 1$, $dy = 0$, and $1 \le x \le 2$.

So $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_1^2 (dx / (x^2 + 1) = \tan^{-1}(x)\Big|_1^2 = \tan^{-1}(2) - \tan^{-1}(1)$

The other three line integrals are similar in that they involve the inverse tangent function and when added together sum to zero. You should verify this result by completing the remaining three line integrals and summing them up.

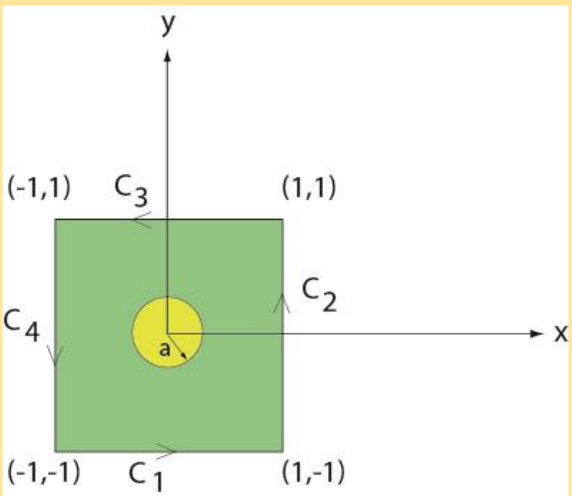
Green's Theorem can be extended for the case where the vector field, **F**(x,y) = P(x,y) **i** + Q(x,y) **j** is not continuous at all points in region R.

Example: Apply Green's Theorem to evaluate the work done by the force vector, **F**, going completely around the square region, R, with vertices at (-1,-1), (-1,1), (1,1), (1,-1) returning to (-1,-1) (shown below) counterclockwise.

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint_R [\partial Q/\partial x - \partial P/\partial y]\, dA$$

where $\mathbf{F} = [\,y/(x^2+y^2)\,]\,\mathbf{i} - [\,x/(x^2+y^2)\,]\,\mathbf{j} = P\,\mathbf{i} + Q\,\mathbf{j}$

As in the previous example $\partial Q/\partial x = (x^2 - y^2)/(x^2+y^2) = \partial P/\partial y$
But these partial derivatives don't exist in the region, R, at the origin (0,0). So "cut out" a small circular region of radius a centered at the origin. Then the partial derivatives exist everywhere in the new region R' which permits using the right hand side. Thus the integral over R' equals zero since $\partial Q/\partial x = \partial P/\partial y$. See the figure below.



But now in the region R' you have you have two curves around its periphery, C and C_o .

Here C consists of C_1, C_2, C_3, and C_4 where C_1 is the portion of C extending from

(-1,-1) to (1,-1). Likewise (moving counterclockwise around the square) C_2 is from

(1,-1) to (1,1), C_3 is from (1,1) to (-1,1), and C_4 is from (-1,1) back to (-1,-1).

C_o is a circular curve of radius a going around the origin also in the counterclockwise direction.

Strategy: Remove the discontinuity in the vector function, F, at the origin by removing the origin from the original region, R. Replace it with a new region, R'. Then apply Green's theorem to the new region, R' . i.e.

$$I = \int_C \mathbf{F}\,.\,d\mathbf{r} - \int_{C_o} \mathbf{F}\,.\,d\mathbf{r} = \iint_{R'} [\partial Q/\partial x - \partial P/\partial y]\, dA = 0$$

Now the vector function, **F**, is continuous in R'. So the partial derivatives exist

and from before $\partial Q/\partial x = \partial P/\partial y$ so that $\partial Q/\partial x - \partial P/\partial y = 0$

Note the negative sign for the path C_o since the region R' is on the right hand side

when traversing C_o in the counterclockwise direction.

So $$\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_{C_o} \mathbf{F}\,.\,d\mathbf{r}$$

$$\mathbf{F}\,.\,d\mathbf{r} = [(y\,\mathbf{i} - x\,\mathbf{j})/(x^2 + y^2)]\,.\,[dx\,\mathbf{i} + dy\,\mathbf{j}]$$

On C_o $x = a\cos\theta$, $y = a\sin\theta$, $dx = -a\sin\theta\, d\theta$, and $dy = a\cos\theta\, d\theta$.

So $\mathbf{F}\,.\,d\mathbf{r} = (-a^2\sin^2\theta - a^2\cos^2\theta)\, d\theta / a^2 = -1\, d\theta$

So $$\int_C \mathbf{F}\,.\,d\mathbf{r} = \int_{C_o} \mathbf{F}\,.\,d\mathbf{r} = -\int_{\theta=0}^{\theta=2\pi} d\theta = -2\pi$$ which is the result.

Basics of Topology

In a Nut Shell: Curves and domains in two and three dimensions can be classified in

different ways that are important when applying Green's Theorem and Stokes Theorem.

The table below provides definitions of terms related to the topology of curves and domains.

Definitions: Let D be the domain

Simple Curve	The curve does not intersect itself
Closed Curve	The terminal point coincides with the initial point of the curve
Open Domain	The domain contains points inside but not on the boundary of the domain
Closed Domain	The domain contains points within and on the boundary of the domain
Connected Domain	Any two points in the domain can be joined by a path within the domain
Simply Connected Domain	Every simple closed curve in the domain encloses only points within the domain