Integration of Rational Functions using Partial Fractions

In a Nut Shell: Recall that a rational function is simply the ratio of two polynomials, say $P(x)$ and $Q(x)$. Let $R(x)$ be the following rational function.

$$\mathbf{R(x) = P(x) / Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials i.e. $P(x) = x^3 + x$ and $Q(x) = x + 1$

The strategy involved in evaluating integrals of rational functions is then to express the rational function as a sum of simpler functions called partial fractions. Thus the method of integration is called the method of "partial fractions".

Note: If $P(x)$ is of equal or higher degree than $Q(x)$ first divide $Q(x)$ into $P(x)$ using long division so that $P(x)$ is of lower degree than $Q(x)$.

The result for the rational function, $R(x)$, is:

$$R(x) = p(x) + P(x)/Q(x) = p(x) + \mathbf{F_1(x) + F_2(x) + \ldots}$$

The $\mathbf{F_i(x)}$, "the partial fractions", commonly have two forms that are useful.

a. $A/(ax + b)^n$

b. $(Bx + C)/(ax^2 + bx + c)^n$

with $b^2 - 4ac < 0$ in the quadratic term

If the exponent, n, is such that n > 1, then the decomposition must account for the multiplicity of these terms. i.e. for forms as in part a and in part b

If $n = 2$, then $P(x)/Q(x) = A_1/(ax + b) + A_2/ax + b)^2$

A similar situation holds for multiplicity of quadratic forms as in part b. i.e. If $n = 2$,

$$P(x)/Q(x) = (B_1(x) + C_1)/(ax^2 + bx + c)$$

$$+ (B_2(x) + C_2)/(ax^2 + bx + c)^2$$

Example Partial Fractions with Distinct Linear Factors

$$I = \int dx / (x^2 + x - 2)$$

Note: $P(x) = 1$ and $Q(x) = x^2 + x - 2$ **So long division is unnecessary.**

Here $P(x)$ is of degree 0 and $Q(x)$ is of degree 2.

$$1/(x^2 + x - 2) = 1/(x - 1)(x + 2) = A/(x - 1) + B/(x + 2)$$

The terms $A/(x - 1)$ and $B/(x + 2)$ are the "partial fractions"

Next put the partial fractions under a common denominator. The result is:

$$1/(x^2 + x - 2) = 1/(x-1)(x+2) = [A(x+2) + B(x-1)]/(x-1)(x+2)$$

and equate numerators on both sides:

$$1 = A(x+2) + B(x-1) = x(A+B) + (2A - B)$$

Solve for A and B. **There are two methods to obtain the solution.**

Method 1: Equate terms of equal powers of x on both sides of the equal sign. The result of this yields two equations in the unknowns A and B as follows:

$2A - B = 1$ and $A + B = 0$ Solve for A and B. $A = 1/3$, $B = -1/3$

Method 2: Both sides of the equation $1 = A(x+2) + B(x-1)$ must hold for any value of x. So pick values of x that enable you to solve most easily for A and B.

For example pick $x = 1$, which gets rid of B. Then you get $1 = 3A$. so $A = 1/3$. Next pick $x = -2$ which gets rid of A. Then $1 = -3B$ and $B = -1/3$.
Note: This method may be quicker and easier.

So $I = (1/3)\int dx/(x-1) - (1/3)\int dx/(x+2)$

and $I = (1/3)\ln|x-1| - (1/3)\ln|x+2| + C$ (result)

Example Partial Fractions where Long Division is required and there are no repeated roots.

$I = \int f(x)\,dx$ where

$$f(x) = P(x)/Q(x) = [2x^3 - 4x^2 - 15x + 5]/[x^2 - 2x - 8]$$

Here P(x) is of degree 3 and Q(x) is of degree 2.
Since the degree of the polynomial in the numerator exceeds that of the denominator, one must use long division to obtain:

$$f(x) = 2x + (x+5)/(x^2 - 2x - 8) = 2x + (x+5)/[(x-4)(x+2)]$$

$$I = \int f(x)\,dx = \int 2x\,dx + \int (x+5)/[(x-4)(x+2)]\,dx$$

Now use partial fractions for the second integral.

$$(x+5)/[(x-4)(x+2)] = A/(x-4) + B/(x+2)$$

The terms $A / (x – 4)$ and $B / (x + 2)$ are the "partial fractions". Next put under the common denominator $(x-4)(x+2)$ and equate both sides

$$(x + 5) / [(x – 4)(x + 2)] = [A(x + 2) + B(x - 4)] / [(x – 4)(x + 2)]$$

So $A(x + 2) + B(x – 4) = x + 5$ which must hold for any value of x.

Use Method 2 picking convenient values of x to solve for A and for B.
Pick $x = 4$. Then $6A = 9$ and $A = 3/2$. Next pick $x = -2$. Then $-6B = 3$ and $B = - ½$.

Then the integrals become

$$I = \int 2x dx + (3/2) \int dx / (x – 4) - (1/2) \int dx / (x + 2)$$

So $I = x^2 + (3/2) \ln|x – 4| - (1/2) \ln|x + 2|$ (result)

Example Partial Fractions with Repeated Roots

$I = \int f(x) dx$ where

$$f(x) = [5x^2 + 20 x + 6] / [x^3 + 2x^2 + x]$$

or $f(x) = [5x^2 + 20 x + 6] / [x(x^2 + 2x + 1)]$

Note: $f(x) = P(x)/Q(x)$ where $P(x) = 5x^2 + 20 x + 6$ and $Q(x) = x^3 + 2x^2 + x$
So long division is unnecessary. Next factor the denominator as follows:

$$f(x) = [5x^2 + 20 x + 6] / [x(x + 1)^2]$$

Note: $Q(x)$ is a product of linear factors x and $(x + 1)$. **Here $(x + 1)$ is repeated.**

Now use partial fractions taking into account the repeated factor for $(x + 1)$.

$$[5x^2 + 20 x + 6] / [x(x + 1)^2] = A/x + B / (x + 1) + C / (x + 1)^2$$

Put this expression under a common denominator which yields,

$$A(x + 1)^2 + Bx(x + 1) + Cx = [5x^2 + 20 x + 6]$$

Use Method 2 pick convenient values of x to solve for A, B, and C using this equation. **Note:** The convenient values of x are 0, -1, and 1.

Pick $x = 0$ (both sides of the equal sign) Then $A = 6$

Next pick $x = -1$. This gives $- C = -9$. So $C = 9$.

Finally pick $x = 1$ and use the calculated values found for A and C.
This gives $6(4) + 2B + 9(1) = 5 + 20 + 6$. Then $2B = -2$ and $B = -1$.

In summary $A = 6$, $B = -1$, $C = 9$

So $\int [A/x]dx + \int [B/(x + 1)]dx + \int [C / (x + 1)^2]\, dx$ becomes

$I = \int 6dx/x - \int dx/(x+1) + 9\int dx/(x + 1)^2$

The third integral can by simplified by letting $u = x + 1$, $du = dx$, $9 \int u^{-2}\, du$

and $I = 6 \ln|x| - \ln|x + 1| - 9 (x + 1)^{-1}$ (result)

Example: $I = \int [\ln(x+1) / x^2]\, dx$

Strategy: First to simplify the integral use integration by parts.

Pick $u = \ln(x+1)$ $dv = dx/x^2$

$du = dx/(x+1)$ $v = - 1/x$

$I = - (1/x) \ln(x+1) + \int dx / [x(x+1)]$

The integral involves a rational function. So apply the method of partial fractions.

$f(x) = 1 / x(x+1)$

$A/x + B/(x+1) = 1 / x(x+1)$

$A(x+1) + Bx = 1$ Pick $x = 0$, then $A = 1$, Pick $x = -1$, then $B = -1$

So $\int dx / [x(x+1)]$ $\int dx/x - \int dx / (x+1)$

Result: $I = - (1/x) \ln(x+1) + \ln(x) - \ln(x+1) + C$