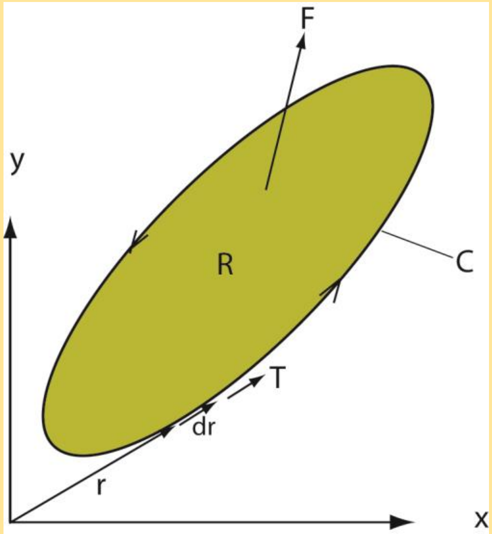
Stokes' Theorem

Background: Recall that Green's theorem gives the relationship between a line integral around a simple closed curve, C, in the x-y plane to a double integral over the plane region R bounded by C. See the figure below.



In its "**curl form**"

Green's Theorem is: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_R \text{curl}_z \mathbf{F}\ \ dA$

Where

$\mathbf{F} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ = the vector field

$d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} = \mathbf{T}\,ds$

$\mathbf{T}$ = the unit tangential vector to the path C

ds = the arc length along the curve, C

dA = the element of area in R

Note: $\text{curl}_z \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = [\partial Q/\partial x - \partial P/\partial y]\,\mathbf{k}$

In other words, $\text{curl}_z \mathbf{F}$ is the z-component of curl $\mathbf{F}$ = curl $\mathbf{F} \cdot \mathbf{k}$

In a Nut Shell: Stokes' Theorem extends the "curl form" of Green's Theorem from two to three dimensions. The line integral of a vector field, **F**, around a simple, closed, piecewise smooth curve, C, with positive orientation equals the curl of the vector field, curl **F**, over the piecewise smooth oriented surface, S, with unit vector **n** .

Stokes' Theorem

$$\int_C \mathbf{F}\,(x,y,z)\,\mathbf{.}\; d\mathbf{r} \;=\; \int\!\!\int_S \text{curl}\,\mathbf{F}\,\mathbf{.}\; d\mathbf{S} \;=\; \int\!\!\int_S \text{curl}\,\mathbf{F}\,\mathbf{.}\,\mathbf{n}\, dS$$

where **F** (x,y,z) is a vector field (could represent a force)

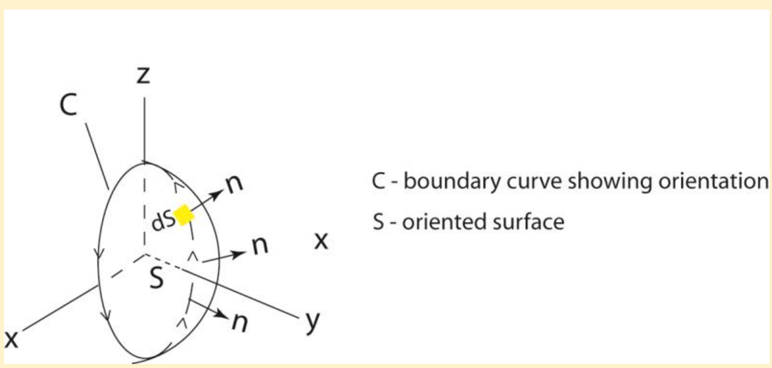
d**r** = the differential vector along the curve, C, in space

d**r** = **T** ds **T** is the unit tangential vector to the curve, C, at any point

ds = the element of arc length along the space curve C

n = the unit normal vector to the surface, S, inside the space curve, C

d**S** = the oriented element of surface area on S



Strategies in the application of Stokes' Theorem

$$\int_C \mathbf{F} \bullet d\mathbf{r} \;=\; \int\!\!\int_S \text{curl}\,\mathbf{F} \bullet \mathbf{dS}$$

F(x,y,z) = vector field, dr = element of arc length on C, d**r** = **T** ds

dS = element of oriented surface area, **dS** = **n** dS, **n** = unit normal to S

Method 1: Evaluate $\int_C \mathbf{F} \bullet d\mathbf{r}$ directly along space curve C

Method 2: Evaluate $\iint_S \text{curl}\,\mathbf{F} \bullet d\mathbf{S}$ directly over the surface S

Method 3A:

Transform the double integral $\iint_S \text{curl}\,\mathbf{F} \bullet d\mathbf{S}$ to a surface integral as follows:

$$\iint_S \text{curl}\,\mathbf{F} \bullet d\mathbf{S} = \iint_S \text{curl}\,\mathbf{F} \bullet \mathbf{n}\, dS$$

where $\mathbf{n} = (\mathbf{r_u} \times \mathbf{r_v}) / |(\mathbf{r_u} \times \mathbf{r_v})|$

$$\iint_S \text{curl}\,\mathbf{F} \bullet (\mathbf{r_u} \times \mathbf{r_v}) / |(\mathbf{r_u} \times \mathbf{r_v})| dS$$

Use $|(\mathbf{r_u} \times \mathbf{r_v})|\, dS = dA$ to obtain the final result

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_R \text{curl}\,\mathbf{F} \bullet (\mathbf{r_u} \times \mathbf{r_v})\, dA$$

where dA is the element of area on R, on the u-v plane

Method 3B:

Transform the double integral $\iint_S \text{curl}\,\mathbf{F} \bullet d\mathbf{S}$ to a surface integral as follows:

$$\iint_S \text{curl}\,\mathbf{F} \bullet d\mathbf{S} = \iint_S \text{curl}\,\mathbf{F} \bullet \mathbf{n}\, dS$$

where $\mathbf{n} = (\mathbf{r_u} \times \mathbf{r_v}) / |(\mathbf{r_u} \times \mathbf{r_v})|$

$$\iint_S \text{curl}\,\mathbf{F} \bullet (\mathbf{r_u} \times \mathbf{r_v}) / |(\mathbf{r_u} \times \mathbf{r_v})| dS$$

Use $|(\mathbf{r_u} \times \mathbf{r_v})|\, dS = dA$ to obtain the final result

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_R \text{curl}\,\mathbf{F} \bullet (\mathbf{r_u} \times \mathbf{r_v})\, dA$$

where dA is the element of area on R, on the u-v plane

Comparison of Green's Theorem and Stokes' Theorem

Green's theorem gives the relationship between a line integral around a simple closed curve, C, in a plane and a double integral over the plane region R bounded by *C*. It is a special two-dimensional case of the more general Stokes' theorem.

Green's theorem in its "**curl form**".

where $\mathbf{F} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$

is as follows: (curl form of Green's Theorem)

$$\int_C \mathbf{F}(x,y) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_R \mathrm{curl}_z\, \mathbf{F}\, dA \qquad \text{equation (1)}$$

where $\mathrm{curl}_z\, \mathbf{F}$ is the z-component of curl $\mathbf{F}$ = curl $\mathbf{F} \cdot \mathbf{k}$

Stokes' Theorem gives the relationship between a line integral around a simple closed curve, C, in space, and a surface integral over a piecewise, smooth surface.

$$\int_C \mathbf{F}(x,y,z) \cdot d\mathbf{r} = \int\int_S \mathrm{curl}\, \mathbf{F} \cdot \mathbf{n}\, dS \qquad \text{equation (2)}$$

where $\mathbf{F}(x,y,z)$ is a vector field (could represent a force)

$d\mathbf{r} = \mathbf{T}\, ds$ **T** is the unit tangential vector to the space curve, C, at any point

ds = the element of arc length along the space curve C

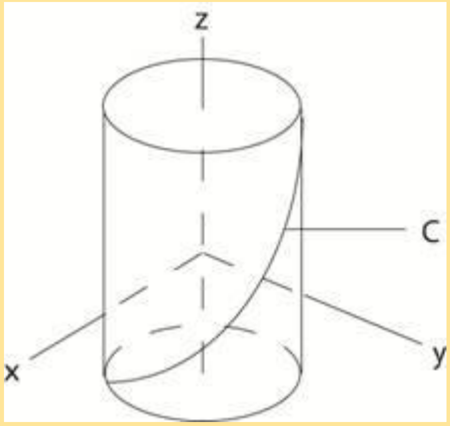
n = the unit normal vector to the surface, S, inside the space curve, C

dS = element of surface area on S

Note the similarities between equations (1) and (2).

In summary, Stokes' Theorem states that the line integral around the boundary curve of S of the tangential component of **F** equals the surface integral of the normal component of the curl of **F**.

Example: Use Stokes' Theorem to evaluate the line integral of the space curve, C, formed by the intersection of the plane $x - 2y + z = 5$ with the cylinder $x^2 + y^2 = 9$



where the vector field is

$$\mathbf{F}(x,y,z) = (x^2 - 3y^2)\, i + (z^2 + y)\, j + (x + 2z^2)\, k$$

Now Stokes' Theorem is $\displaystyle\int_C \mathbf{F}(x,y,z)\,.\,d\mathbf{r} = \int\int_R \text{curl}\,\mathbf{F}\,.\,\mathbf{n}\; dS$

$$\text{curl}\,\mathbf{F} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x^2 - 3y^2) & (z^2 + y) & (x + 2z^2) \end{vmatrix}$$

$$\text{curl}\,\mathbf{F} = [-2z]\,\mathbf{i} - [1]\,\mathbf{j} + [6y]\,\mathbf{k}$$

The surface integral is over the plane of intersection, so that the unit normal to the plane, **n**, can be determined using the gradient to the plane. i.e.

$$\mathbf{n} = \text{grad}(f)/|\,\text{grad}(f)| =$$

$$[\partial f/\partial x\,\mathbf{i} + \partial f/\partial y\,\mathbf{j} + \partial f/\partial z\,\mathbf{k}]/[\sqrt{[(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + (\partial f/\partial z)^2]}$$

where $f(x,y,z) = x - 2y + z - 5 = 0$

so $\partial f/\partial x = 1, \quad \partial f/\partial y = -2, \quad \partial f/\partial z = 1$ and

$\mathbf{n} = [\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}\,] / \sqrt{6}$ **Note:** **n** is the normal to the slanted plane of intersection

Recall Stokes' Theorem: $\displaystyle\int_C \mathbf{F}(x,y,z)\,.\,d\mathbf{r} = \int\int_R \text{curl}\,\mathbf{F}\,.\,\mathbf{n}\; dS$

$$\text{curl } \mathbf{F} \,.\, \mathbf{n} = \{[-2z]\,\mathbf{i} - [1]\,\mathbf{j} + [6y]\,\mathbf{k}\} \,.\, [\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}] / \sqrt{6}\}$$

$$= [-2z + 2 + 6y] / (1/\sqrt{6})$$

So the surface integral becomes $\int\int_R (1/\sqrt{6})[-2z + 2 + 6y]\, dS$.

where R is the slanted elliptical surface formed by the intersection of the plane and the cylinder. Since this surface integral is difficult to evaluate one strategy is to apply the transformation strategy for surface integrals to obtain an area integral over the circular area in the x-y plane. Call it D.

On the slanted plane dS = element of surface area in R,
dA = element of are in region, D in the xy - plane

$$z = 5 - x + 2y, \quad \text{and} \quad dS = \sqrt{[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1]}\, dA$$

$$\text{so} \qquad dS = \sqrt{[1 + 4 + 1]}\, dA = \sqrt{6}\, dA = \sqrt{6}\, dx\, dy$$

and the area integral becomes $\int\int_D \sqrt{6}\,(1/\sqrt{6})[-2(5 - x + 2y) + 2 + 6y]\, dx\, dy$

$$\int\int_D [-10 + 2x - 4y + 2 + 6y]\, dx\, dy = \int\int_D [-8 + 2x + 2y]\, dx\, dy$$

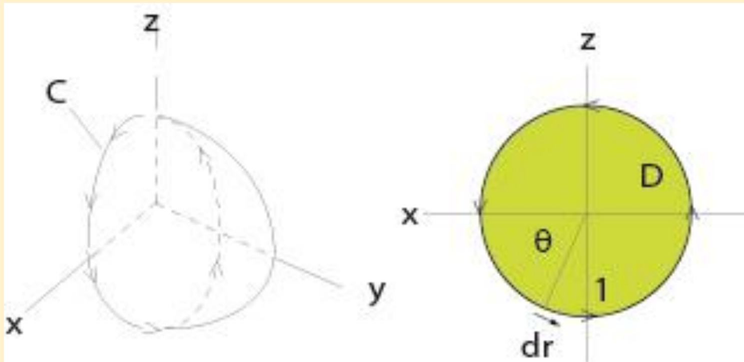
Switch to polar coordinates (D is a circle in the x-y plane of radius 3)

$$\text{where} \quad dA = dx\, dy = r\, dr\, d\theta$$

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} [-8 + 2r\cos\theta + 2r\sin\theta]\, r\, dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} [-8\, r^2/2 + (2\, r^3/3)\cos\theta + (2\, r^3/3)\sin\theta] \Big|_{r=0}^{r=3} d\theta$$

$$\int_{\theta=0}^{\theta=2\pi} [-36 + 18\cos\theta + 18\sin\theta]\, d\theta = -36(2\pi) = -72\pi \qquad \text{(result)}$$

Example: Use Stokes' Theorem to evaluate the line integral of the space curve, C, determined by the intersection of the xz-plane with the hemisphere, S, given by $x^2 + y^2 + z^2 = 1,\ y \geq 0$ oriented in the positive y-axis and the vector field is $\mathbf{F}(x,y,z) = < y, z, x >$. See the figure below.



Method 1 Evaluate the line integral $\int_C \mathbf{F}(x,y,z)\,.\,d\mathbf{r}$ directly.

Note on C $x^2 + z^2 = 1$ and $y = 0$. Define parameters on C: $x = \cos\theta$, $z = -\sin\theta$

So $d\mathbf{r} = -\sin\theta\, d\theta\, \mathbf{i} - \cos\theta\, d\theta\, \mathbf{k}$ and $\mathbf{F} = z\,\mathbf{j} + x\,\mathbf{k}$ giving

$\mathbf{F} \bullet d\mathbf{r} = -\cos^2\theta\, d\theta$ where $0 \le \theta \le 2\pi$

$$\int_0^{2\pi} \mathbf{F} \bullet d\mathbf{r} = -\int_0^{2\pi} \cos^2\theta\, d\theta = -\int_0^{2\pi} [\,(1 + \cos 2\theta)/2\,]\, d\theta = -\pi \text{ (result)}$$

Method 2 Direct evaluation of $\iint_S \text{curl}\,\mathbf{F} \bullet d\mathbf{S} = \iint \text{curl}\,\mathbf{F} \bullet \mathbf{n}\, dS$

$$\text{curl}\,\mathbf{F} = \det \begin{matrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{matrix}$$ where det represents the determinant

$$\text{curl}\,\mathbf{F} = -\,\mathbf{i} - \mathbf{j} - \mathbf{k}$$

Next determine the unit normal vector, **n**, to dS. Let g(x,y,z) be the surface of the hemisphere, S. Then $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$. The unit normal to dS then is $\mathbf{n} = \text{grad } g / |\text{grad } g| = \langle 2x, 2y, 2z \rangle / \sqrt{[(2x)^2 + (2y)^2 + (2z)^2]}$

now $x^2 + y^2 + z^2 = 1$ so $\mathbf{n} = \langle x, y, z \rangle$

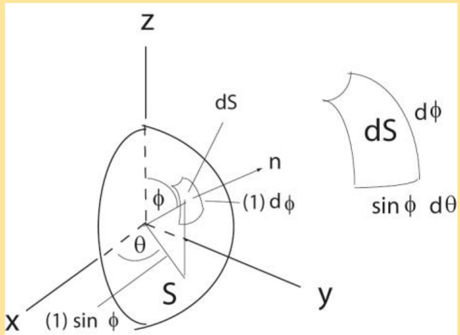
Since the surface, S, is on the hemisphere, switch to spherical coordinates.

Recall $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho\cos\varphi$ and on S $\rho = 1$

So the unit normal becomes $\mathbf{n} = \langle \sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi \rangle$

$\text{curl}\,\mathbf{F}\,.\,\mathbf{n} = -\langle \sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi \rangle$

Next determine the expression for the element of surface area, dS.
Evaluate the expression for the element of surface area, dS, using the figure below.



Since the radius of the hemisphere is one, dS = sin φ dφ dθ

curl F • **n** dS = **-** [sin φ cos θ + sin φ sin θ + cos φ] sin φ dφ dθ

$$\int\int \text{curl } \mathbf{F} \bullet \mathbf{n}\text{) dS} = -\int_0^{\pi}\int_0^{\pi} [\ \sin\varphi\cos\theta + \sin\varphi\sin\theta + \cos\varphi\ ]\sin\varphi\, d\varphi\, d\theta$$

$$\int\int \text{curl } \mathbf{F} \bullet \mathbf{n}\text{) dS} = -\int_0^{\pi}\int_0^{\pi} [\ \sin^2\varphi\cos\theta + \sin^2\varphi\sin\theta + \sin\varphi\cos\varphi\ ]\, d\varphi\, d\theta$$

$$\int\int \text{curl } \mathbf{F} \bullet \mathbf{n}\text{) dS} = -\int_0^{\pi}\int_0^{\pi} [\ \sin^2\varphi\cos\theta + \sin^2\varphi\sin\theta + \sin\varphi\cos\varphi\ ]\, d\theta\, d\varphi$$

$$\int\int \text{curl } \mathbf{F} \bullet \mathbf{n}\text{) dS} = \int_0^{\pi} [\ -\sin^2\varphi\sin\theta + \sin^2\varphi\cos\theta - (\ \tfrac{1}{2}\sin 2\varphi\ )\theta\ \Big|_0^{\pi}\ d\varphi$$

$$\int\int \text{curl } \mathbf{F} \bullet \mathbf{n}\text{) dS} = \int_1^{\pi} [-\sin^2\varphi\,(0-0) + (-1-1)\sin^2\varphi + (\pi/2)\sin 2\varphi\ ]\ d\varphi$$

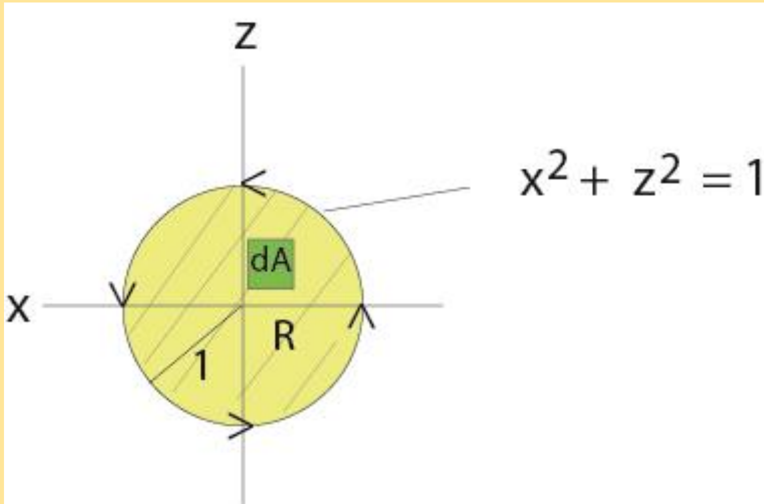
$$\int\int \text{curl } \mathbf{F} \bullet \mathbf{n}\text{) dS} = \int_0^{\pi} [(-2\{(\ 1 + \cos 2\varphi)/2\} + (\pi/2)\sin 2\varphi\ ]\ d\varphi$$

$$\int\int \text{curl } \mathbf{F} \bullet \mathbf{n}\text{) dS} = [-\ 2(\pi/2) + \ 2\sin 2\varphi\ \Big|_0^{\pi} = -\pi \qquad \text{(same result)}$$

Method 3 Change $\int\int$ (curl **F** • **n**) dS into a surface integral .

$$\int\int_S \text{curl } \mathbf{F} \bullet \mathbf{n}\ dS = \int\int_R (\text{ curl } \mathbf{F} \bullet (\mathbf{r_u} \times \mathbf{r_v})\ dA$$

The transformed surface from S to R is shown below.



On S $\quad \mathbf{r} = < x , y , z > = < x, \sqrt{ (1 – x^2 – z^2) } , z >$

$\mathbf{r}_x = < 1, -x / \sqrt{(1 – x^2 – z^2)}, 0 > , \quad \mathbf{r}_z = < 0, -z / \sqrt{(1 – x^2 – z^2)}, 1 >$

Since positive orientation is in the positive y-direction dS = ($\mathbf{r}_z$ x $\mathbf{r}_x$) dA

$$\mathbf{r}_z \times \mathbf{r}_x = \det \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -z / \sqrt{(1 – x^2 – z^2)} & 1 \\ 1 & -x / \sqrt{(1 – x^2 – z^2)} & 0 \end{array}$$

Method 3 Changing ∫ ∫ curl **F** ● **n**) dS into a surface integral (continued)

$$\int_S \int (\text{curl } \mathbf{F} \bullet \mathbf{n}) \, dS = \int_R \int (\text{curl } \mathbf{F} \bullet (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

$$\mathbf{r}_z \times \mathbf{r}_x = \det \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -z / \sqrt{(1 – x^2 – z^2)} & 1 \\ 1 & -x / \sqrt{(1 – x^2 – z^2)} & 0 \end{array}$$

$\mathbf{r}_z \times \mathbf{r}_x = < -x / \sqrt{(1 – x^2 – z^2)} , 1, -z / \sqrt{(1 – x^2 – z^2)} >$

recall curl **F** = **-** < 1, 1, 1 > so

$\text{curl } \mathbf{F} \bullet (\mathbf{r}_z \times \mathbf{r}_x) = -x / \sqrt{(1 – x^2 – z^2)} - 1 - z / \sqrt{(1 – x^2 – z^2)}$

Since the transformed surface, R, in the x-z plane is a circular area, it is convenient to switch to polar coordinates. So $x = x(r,\theta)$, $z = z(r,\theta)$ as follows:

$$x = r\cos\theta, \qquad z = -\,r\sin\theta \quad \text{and} \quad dA = r\,dr\,d\theta$$

The resulting surface integral is

$$\int_{r=0}^{r=1} \int_{\theta=0}^{\theta=2\pi} -\,r\cos\theta / \sqrt{(1-r^2)} \;-1\; + \; r\sin\theta / \sqrt{(1-r^2)}\;]\;d\theta\;\; r\,dr$$

$$\int_{r=0}^{r=1} \left[\; -\,r\sin\theta / \sqrt{(1-r^2)} \;-\theta\; + \; r\cos\theta / \sqrt{(1-r^2)}\;\right]_0^{2\pi} \; r\,dr$$

$$\int_{r=0}^{r=1} (-2\pi)\; r\,dr \;=\; -2\pi\,(1/2) \;=\; -\,\pi \qquad\qquad \text{(same result)}$$