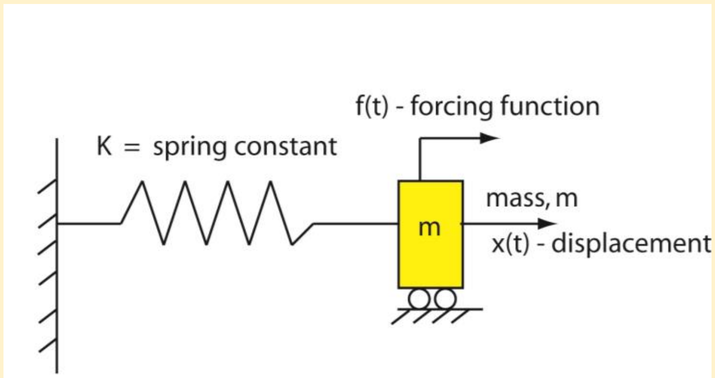
Steady Periodic Vibrations

In a Nut Shell: Displacement, x(t), of an undamped mechanical system with mass, m, and spring constant, k , (figure below) undergoes steady periodic vibrations when subjected to a periodic forcing function, f(t).



The steady state response, (also called the steady periodic solution) is also the particular solution) , $x_p(t)$. It is governed by the forcing function since there is no damping to diminish the response. The differential equation of motion is:

$$m\,x'' + k\,x = f(t) \quad \text{------------------------------------} \quad (1)$$

where m = mass of the system
$x'' = d^2x/dt^2$ = the acceleration of the mass, m
k = spring constant
x(t) = displacement of the mass
f(t) = periodic forcing function

Representation of Forcing Function, f(t)

Use Fourier Series as a way to represent more complicated (more realistic) forcing functions, f(t), with direct applications to vibration problems. Suppose f(t) is represented as follows:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t/L) \quad \text{------------------------------------} \quad (2)$$

where b_n are the Fourier coefficients **(need to be determined)**

Strategy to find the steady periodic response, x(t), for the differential equation governing the mechanical system describe in the equations below

$$m\,x'' + k\,x = f(t) \quad \text{--} \quad (1)$$

where the general, periodic forcing function is represented by

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t/L) \quad \text{--------------------------------------} \quad (2)$$

involves four steps.

Step 1: Find the Fourier coefficients for f(t), of period 2L, as follows:

$$b_n = (2/L)\int_0^L f(t)\sin(n\pi t/L)\,dt \qquad \text{and substitute into eq. (2)}$$

Step 2: Assume a steady periodic response of the mechanical system governed by eq. (1), $x_{sp}(t)$ as follows:

$$x_{sp}(t) = \sum_{n=1}^{\infty} c_n \sin(n\pi t/L) \quad \text{------------------------------------} \quad (3)$$

where c_n are unknown coefficients (yet to be determined)

Step 3: Substitute (3) into the equation of motion, eq. (1).

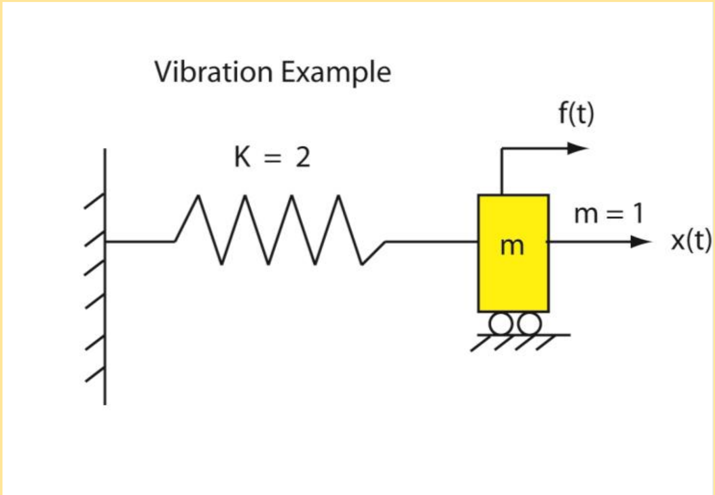
Step 4: Equate coefficients of $\sin(n\pi t/L)$ and solve for c_n using the values obtained for b_n.

Steady, periodic response of the mechanical system is then given by:

$$x_{sp}(t) = \sum_{n=1}^{\infty} c_n \sin(n\pi t/L) \qquad \text{(result)}$$

Example: Find the steady periodic response, $x_{sp}(t)$, for the spring-mass system shown in the figure below subject to the prescribed forcing function, f(t).

$$x'' + 2\,x = f(t) \quad \text{--} \quad (1)$$



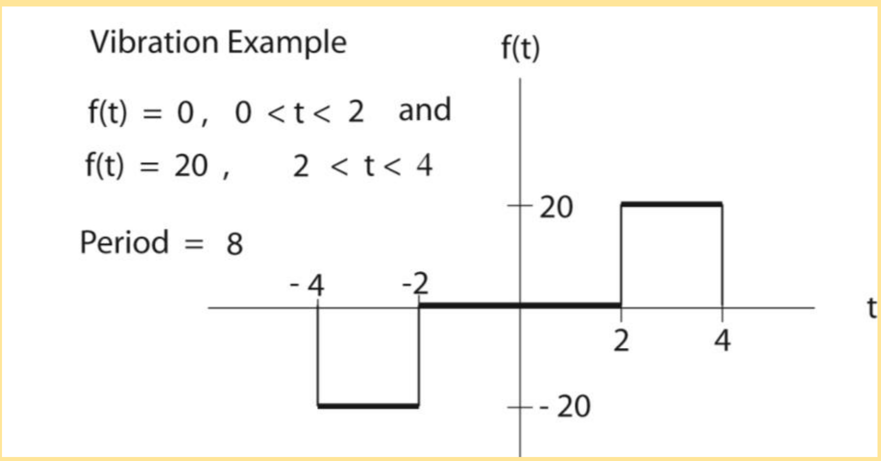
The periodic forcing function, f(t), is an "odd" function of period 8 given by

$$f(t) = \begin{cases} 0 & \text{for } 0 < t < 2 \\ 20 & \text{for } 2 < t < 4 \end{cases}$$

See the figure below.

where

$$f(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t/4) \quad \text{-----------------------------------} \quad (2)$$



Strategy: Apply the four step method.

Step 1: Find the Fourier coefficients for f(t) as follows:

Recall

$$b_n = (2/L) \int_0^L f(t) \sin(n\pi t/L)\, dt$$

So

$$b_n = (2/4) \int_0^4 f(t) \sin(n\pi t/4)\, dt = (1/2)) \int_2^4 20 \sin(n\pi t/4)\, dt$$

$$b_n = -10\,(4/n\pi) \cos(n\pi t/4) \Big|_2^4 = -40/n\pi\pi)[\cos n\pi - \cos(n\pi/2)]$$

Step 2: Assume a steady periodic response, $x_{sp}(t)$ as follows:

$$x_{sp}(t) = \sum_{n=1}^{\infty} c_n \sin(n\pi t/4) \quad \text{------------------------------------} \quad (3)$$

Step 3: Put $x_{sp}(t)$ into eq. of motion (eq 1) $x'' + 2x = f(t)$

$$\sum_{n=1}^{\infty} [-(n\pi/4)^2 + 2]\, c_n \sin(n\pi t/4) = f(t)$$

Step 4: Equate coefficients of $\sin(n\pi t/4)$ and solve for c_n using the values obtained for b_n.

$$[-(n\pi/4)^2 + 2]\, c_n = -40/n\pi\pi)[\cos n\pi - \cos(n\pi/2)]$$

$$c_n = -40/n\pi\pi)[\cos n\pi - \cos(n\pi/2)] / [2-(n\pi/4)^2]$$

The resulting steady, periodic response is:

$$x_{sp}(t) = \sum_{n=1}^{\infty} c_n \sin(n\pi t/4) \quad \text{is the steady periodic response} \qquad \text{(result)}$$

where $c_n = -40/n\pi\pi)[\cos n\pi - \cos(n\pi/2)] / [2-(n\pi/4)^2]$