1-D Wave Equation/String Vibrations

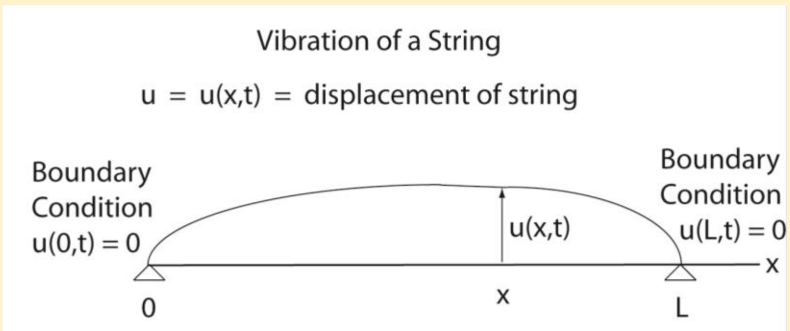
In a Nut Shell: The 1-D wave equation (string vibration, see figure below) is governed by the following partial differential equation:

$$\partial^2 u/\partial t^2 \;=\; a^2\, \partial^2 u/\partial x^2 \quad \text{---------------------} \quad (1)$$

where u = u(x,t) = the displacement of the string
x = the position along the string
t = the time that the displacement occurs at x

and

a is the speed of wave propagation (material constant)



Note: Since the partial differential equation is second order in both its derivatives with respect to x and t you **need two boundary conditions and two initial conditions**.

The desired outcome is to predict the displacement of the string, u(x,t), subject to its boundary and initial conditions (provided in the tables below).

The common **boundary conditions** at the ends of the string are as follows:

a. Fixed end at x = 0 — $u(0,t) = 0$
b. Fixed end x = L — $u(L,t) = 0$
c. Sliding end condition at x = 0 — $\partial u(0,t)/\partial x = 0$
d. Sliding end condition at x = L — $\partial u(L,t)/\partial x = 0$

or any combination of these boundary conditions

The common **initial conditions** at the ends of the string are as follows:

a. Prescribed displacement at t = 0 — $u(x,0) = f(x)$
b. Prescribed speed of string at t = 0 — $\partial u(x,0)/\partial t = g(x)$

Strategy: A general approach to solving the 1-D wave equation

$$\partial^2 u/\partial t^2 = a^2 \, \partial^2 u/\partial x^2$$

is to assume separation of variables. i.e. Assume:

$$u(x,t) = X(x)\, T(t)$$

Substitution of $u(x,t) = X(x)\, T(t)$ into the wave equation gives

$$X(x)\, dT^2/dt^2 = a^2\, d^2X/dx^2$$

Then by division $(d^2X/dx^2)/X = (dT^2/dt^2)/a^2T = -\lambda$ = separation constant

So $d^2X/dx^2 + \lambda X = 0$ and $dT^2/dt^2 + \lambda a^2 T = 0$

Note: To solve the 1-D wave equation you need to solve two eigenvalue problems.

Further note that the separation constant could be zero, negative, or positive.

Examine each case separately.

Solution of eigenvalue problem. **Strategy:** Start with the eigenvalue problem for X(x).

$$d^2X/dx^2 + \lambda X = 0 \quad \text{subject to the boundary conditions}$$

$$X(0) = X(L) = 0$$

And consider each case for λ separately.

The cases are $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

The result of this calculation yields the eigenvalues λ_n and eigenvectors, $X_n(x)$.

Strategy: Substitute the eigenvalues, λ_n , into the equation for T(t) to obtain

$$dT^2/dt^2 + \lambda_n a^2 T = 0$$

Solution of this equation normally yields $T_n(t) = C_n \cos a\sqrt{\lambda_n} t + D_n \sin a\sqrt{\lambda_n} t$

Then combine with $X_n(x)$ with $T_n(t)$ to obtain the product solution $T_n(t)\, X_n(x)$.

$$u_n(x,t) = [C_n \cos a\sqrt{\lambda_n} t + D_n \sin a\sqrt{\lambda_n} t\,]\, X_n(x).$$

Now sum up each of the terms $u_n(x,t)$ to obtain the solution for $u(x,t)$.

$$u(x,t) = \sum_{n=1}^{\infty} [C_n \cos a\sqrt{\lambda_n} t + D_n \sin a\sqrt{\lambda_n} t\,] X_n(x).$$

Note: The solution for the displacement of the string, $u(x,t)$, involves a Fourier series.

Strategy: Determine the Fourier coefficients C_n and D_n from the prescribed initial conditions.

$$u(x,0) = f(x) \quad \text{and} \quad \partial u(x,0)/\partial t = g(x)$$

Note: C_n is determined from $u(x,0)$ and D_n is determined from $\partial u(x,0)/\partial t$.

Example: Find the displacement, $u(x,t)$, of the vibrating string given by the 1-D wave equation

$$\partial^2 u/\partial t^2 = 4\, \partial^2 u/\partial x^2 \qquad 0 < x < \pi, \quad t > 0$$

Subject to the boundary conditions $u(0,t) = u(\pi,t) = 0$

Along with the initial conditions $u(x,0) = \sin x$ and $\partial u(x,0)/\partial t = 1$

Strategy: Start by assuming separation of variables

$$u(x,t) = X(x)\, T(t)$$

Substitution of this expression into the above wave equation gives

$$X(x)\, dT^2/dt^2 = 4\, d^2X/dx^2$$

Then by division $(d^2X/dx^2)/X = (dT^2/dt^2)/\, 4T = -\lambda =$ separation constant

So $d^2X/dx^2 + \lambda X = 0$ and $dT^2/dt^2 + 4\lambda T = 0$

Start with the eigenvalue problem for $X(x)$.

$d^2X/dx^2 + \lambda X = 0$ subject to the boundary conditions

$$X(0) = X(\pi) = 0$$

Case 1 **$\lambda = 0$** $d^2X/dx^2 = 0$ or $X(x) = Ax + B$

$X(0) = 0 = B$ and $X(\pi) = A\pi = 0$ so $A = 0$ There are no eigenvalues for this case.

Case 2 **$\lambda < 0$** Let $\lambda = -\alpha^2$, $\alpha > 0$

$$d^2X/dx^2 - \alpha^2 X = 0$$

So $X(x) = A \cosh \alpha x + B \sinh \alpha x$ and

$X(0) = 0 = A$, $X(\pi) = 0 = B \sinh \alpha\pi$ now $\sinh \alpha\pi \neq 0$ so $B = 0$

Result: No eigenvalues for this case.

Case 3 **$\lambda > 0$** Let $\lambda = \alpha^2$, $\alpha > 0$

$$d^2X/dx^2 + \alpha^2 X = 0$$

So $X(x) = C \cos \alpha x + D \sin \alpha x$ and

$X(0) = 0 = C$, $X(\pi) = 0 = B \sin \alpha\pi$

Now for a nontrivial solution $B \neq 0$ so $\sin \alpha\pi = 0$

which gives $\alpha_n \pi = n\pi$ or $\alpha_n = n$

Result: Eigenvalues for this case $\lambda_n = n^2$ and eigenfunctions, $X_n(x)$ are $\sin nx$

Strategy: Apply these eigenvalues to

$$dT^2/dt^2 + \lambda_n a^2 T = 0 \quad \text{or in this case} \quad dT^2/dt^2 + 4 n^2 T = 0$$

Solution of this equation is $T_n(t) = C_n \cos 2nt + D_n \sin 2nt$

Strategy: Combine $X_n(x,t)$ and $T_n(t)$ to obtain $u_n(x,t)$ which gives

$$u_n(x,t) = [C_n \cos 2nt + D_n \sin 2nt] \sin nx$$

Sum up individual terms to obtain

$$u(x,t) = \sum_{n=1}^{\infty} [C_n \cos 2nt + D_n \sin 2nt] \sin nx$$

Determine the values of C_n and D_n from the prescribed initial conditions.

$$u(x,0) = \sin x = \sum_{n=1}^{\infty} [C_n] \sin nx$$ In this case $n = 1$ which gives $C_1 = 1$

All other C's are zero. Next take the derivative of $u(x,t)$ with respect to time to obtain

$$\partial u(x,t)/\partial t = \sum_{n=1}^{\infty} [- 2n C_n \sin 2nt + 2n D_n \cos 2nt] \sin nx$$

and

$$\partial u(x,0)/\partial t = \sum_{n=1}^{\infty} [2n D_n] \sin nx = 1$$

Now for the initial condition $\partial u(x,0)/\partial t$

$$\sum_{n=1}^{\infty} [2n D_n] \sin nx = 1$$

Strategy: Evaluate the Fourier coefficients, D_n.

$$2n D_n = 2/\pi) \int_{x=0}^{x=\pi} (1) \sin nx \, dx = - (2/\pi) \cos nx \Big|_0^{\pi} = - (2/\pi) [\cos n\pi - 1)$$

or $$2n D_n = 4/n\pi \text{ where } n \text{ is odd}$$

Next solve for D_n $$D_n = 2 / n^2\pi$$

Finally substitute C_n and D_n into

$$u(x,t) = \sum_{n=1}^{\infty} [C_n \cos 2nt + D_n \sin 2nt] \sin nx$$

to obtain

$$u(x,t) = \cos 2t \sin nx + \sum_{n = odd}^{\infty} [(2 / n^2\pi) \sin 2nt] \sin nx$$

which is the displacement of the string at any position, x, at time t. (result)