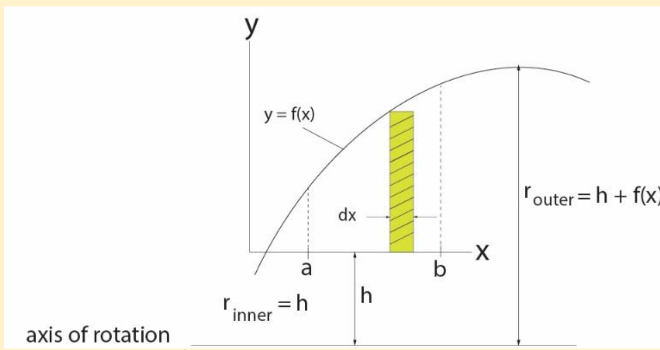
Volume of Revolution using Disks and Shells

In a Nut Shell: There are two methods for calculating volumes of revolution - the method of disks and the method of shells. Calculation of the volume generated by rotating a curve or a set of curves about a designated axis of rotation is a three-step process such as:

For the Method of Disks

Step 1: Identify the element of volume, dV, and show it on the graph of y = f(x)
For a disk element $dV = (\pi r^2_{outer} - \pi r^2_{inner})\, dx$. See figure below.
Step 2: Determine the limits of integration for the region (volume to be calculated)
Step 3: Evaluate the integral

Step 1



Region to be rotated is bounded by x = a, x = b, y = 0, and y = f(x).

Note that the disk (donut) is generated by rotating the hatched area shown about the axis of rotation.

Steps 2 and 3

Establish limits of integration. In this case the integration is from x = a to x = b.

Express the total volume as an integral over the region rotated about the axis of rotation. The disk is defined by $dV = (\pi r^2_{outer} - \pi r^2_{inner})\, dx$ or in this case

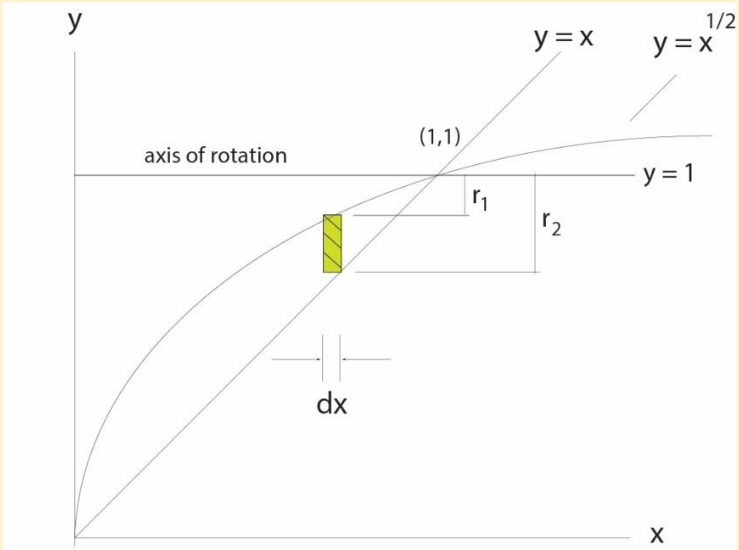
$$dV = [\, \pi (h + f(x))^2 - \pi h^2]\, dx.$$

$$V = \int_{x=a}^{x=b} [\, \pi (h + f(x)^2) - \pi h^2]\, dx$$

Example: Use the method of disks to calculate the volume generated by rotating the curves y = x and $y = \sqrt{x}$ about the line y = 1. Recall the three-step process.

Step 1: Identify the element of volume, dV, and show it on the graph of y = f(x)
Step 2: Determine the limits of integration for the region (volume to be calculated)
Step 3: Evaluate the integral

Step 1 The element of volume, $dV = (\pi r_2^2 - \pi r_1^2)\, dx$.



Note that the "donut" is generated by rotating the hatched area about the axis of rotation.

Here $r_1 = 1 - y$ on the curve $y = \sqrt{x}$ and $r_2 = 1 - y$ on the line $y = x$.

So $r_1 = 1 - \sqrt{x}$ and $r_2 = 1 - x$

Step 2 Limits of integration are from $x = 0$ to $x = 1$.

Step 3

$$V = \int_{x=0}^{x=1} (\pi [(1 - \sqrt{x})^2) - (1 - x)^2]\, dx$$

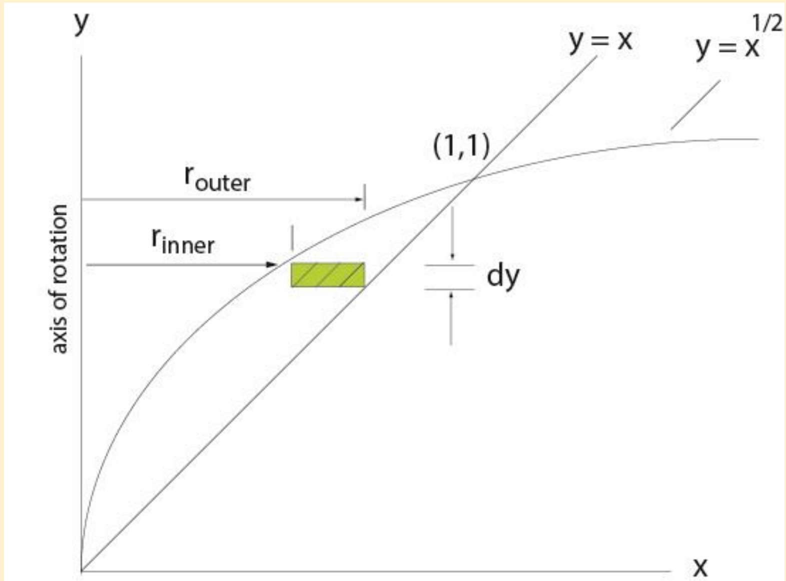
$$V = \pi \int_{x=0}^{x=1} ([(x + 1 - 2\sqrt{x} - x^2 + 2x - 1]\, dx = \pi / 6 \text{ (result)}$$

Example: Use the method of disks to calculate the volume generated by rotating the curves $y = x$ and $y = \sqrt{x}$ about the line $x = 0$. Recall the three-step process.

Step 1: Identify the element of volume, dV, and show it on the graph of y(x)
Step 2: Determine the limits of integration for the region (volume to be calculated)
Step 3: Evaluate the integral

Step 1 The element of volume, $dV = \pi\,(r^2_{outer} - r^2_{inner})\,dy = \pi\,(y^2 - y^4)\,dy$

Note: $r_{outer} = x = y$, $r_{inner} = x = y^2$



Note that the donut is generated by rotation of the hatched area about the axis of rotation. The donut has an outer radius (in terms of y) equal to y. The inner radius (in terms of y) is y^2.

Step 2 Limits of integration are from $y = 0$ to $y = 1$.

Step 3 $r_{outer} = x = y$, $r_{inner} = x = y^2$

$$V = \int_{y=0}^{y=1} \pi\,[\,y^2 - y^4]\,dy = 2\,\pi\,/\,15 \text{ (result)}$$

Volumes of Revolution using the Method of Shells

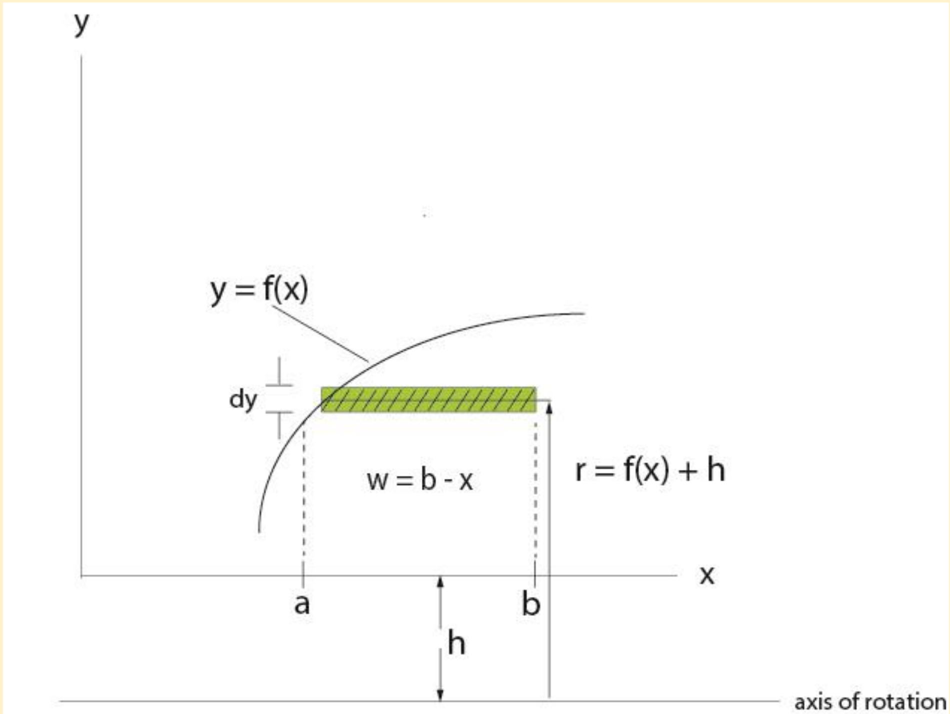
An Alternative to using the Method of Disks

In a Nut Shell: Calculation of the volume generated by rotating a curve or a set of curves about a designated axis of rotation is a three-step process such as:

For the Method of Shells

Step 1: Identify the element of volume, dV, and show it on the graph of y(x)
For a shell element $dV = 2\pi r\,w\,dy$. See figure below.
Step 2: Determine the limits of integration for the region (volume to be calculated)
Step 3: Evaluate the integral

Step 1



Note that the shell is generated by rotating the hatched area shown about the axis of rotation.

Steps 2 and 3

Establish limits of integration. In this case the integration is from y = f(a) to y = f(b).

Express the total volume as an integral over the region rotated about the axis of rotation. The disk is defined by $dV = [\, 2\pi\, r\, w]\, dy$ where $r = f(x) + h$ and $w = b - x = b - g(y)$

$$V = \int_{y=f(a)}^{y=f(b)} [\, 2\pi\, (\, f(x) + h\,)\, (\, b - x)\,]\, dy \qquad \text{where } x = g(y)$$

Example: Calculate the volume generated by rotating the curves y = x and y = √ x about The line y = 1. Recall the three-step process.

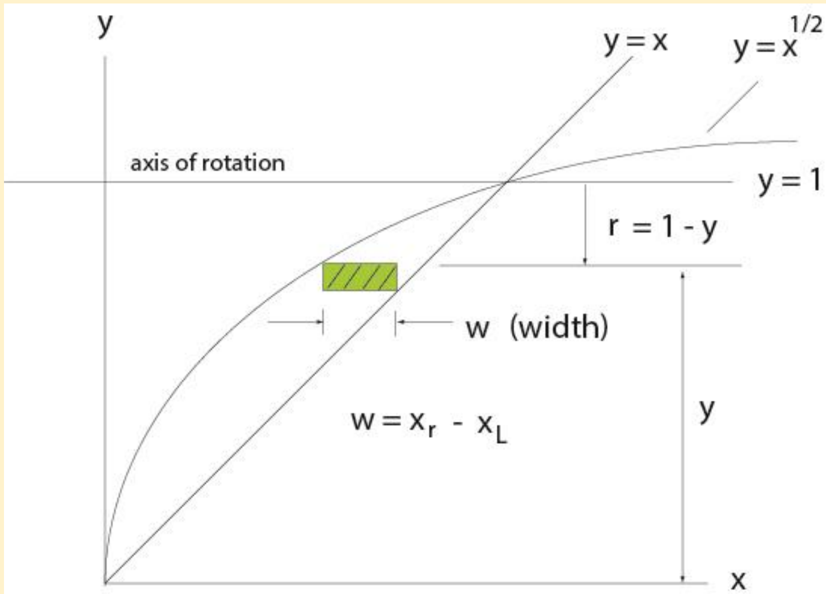
Step 1: Identify the element of volume, dV, and show it on the graph of y(x)

Step 2: Determine the limits of integration for the region (volume to be calculated)

Step 3: Evaluate the integral

Step 1 The element of volume, $dV = 2\pi r (x_r - x_L)\,dy = 2\pi (1 - y)(y - y^2)\,dy$

Note: $r = 1 - y$, $w = x_r - x_L$ where $x_r = y$, $x_L = y^2$



Note that the shell is generated by rotation of the hatched area about the axis of rotation.

Step 2 Limits of integration are from $y = 0$ to $y = 1$.

Step 3

$$V = \int_{y=0}^{y=1} 2\pi\,[(1 - y)(y - y^2)]\,dy$$

$$V = 2\pi \int_{y=0}^{y=1} ([\,y - y^2 - y^2 + y^3]\,dy = \pi / 6 \text{ (result)}$$

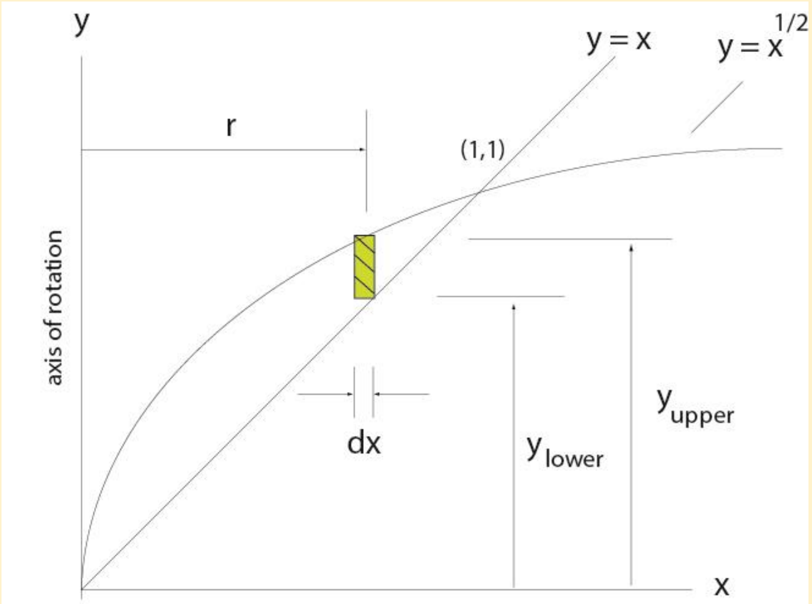
Example: Use the method of shells to calculate the volume generated by rotating the curves $y = x$ and $y = \sqrt{x}$ about the line $x = 0$. Recall the three-step process.

Step 1: Identify the element of volume, dV, and show it on the graph of $y = f(x)$

Step 2: Determine the limits of integration for the region (volume to be calculated)

Step 3: Evaluate the integral

Step 1 The element of volume, $dV = (2\pi r)(y_{upper} - y_{lower})\, dx$.



Note that the "shell" is generated by rotating the hatched area about the axis of rotation.

Step 2 Limits of integration are from $x = 0$ to $x = 1$.

Step 3 Here $r = x$, $y_{upper} = \sqrt{x}$ and $y_{lower} = x$

$$V = \int_{x=0}^{x=1} (2\pi x [\sqrt{x} - x]\, dx$$

$$V = 2\pi \int_{x=0}^{x=1} ([x^{3/2} - x^2]\, dx = 2\pi / 15 \quad \text{(result)}$$