Summary of the Calculus of Vector Fields

In a Nut Shell: There are several helpful theorems in multi-variable calculus of vector fields. They enable you to pick alternative ways to evaluate line integrals and surface integrals. It's best to select the easiest option for your problem.

A **Line Integral**, I_L, is used to evaluate the value of a vector function, **F**(x,y), along a plane curve, C, or of a vector function, **F**(x,y,z), along a space curve, C.

$$I_L = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot (d\mathbf{r}/dt)\, dt$$

The **Surface Integral**, I_s, is analogous to the line integral in that it provides the value of a function, f(x,y,z), evaluated over a "smooth" surface, S, in space. Here the surface integral is

$$I_s = \int\int_S f(x, y, z)\, dS$$

where dS is the element of surface area on the spatial surface.

Green's Theorem gives the relationship between a line integral around a simple closed curve in a plane, C, and a double integral over the enclosed plane region R bounded by C.

$$\int_C P\,dx + Q\,dy = \int\int_R [\partial Q/\partial x - \partial P/\partial y]\, dA \quad \text{(standard form of Green's Theorem)}$$

The curve, C, is said to be positively oriented when traveling counterclockwise around C keeping the region, R, enclosed to the left. If $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$, then $\mathbf{F} \cdot d\mathbf{r} = Pdx + Qdy$

So $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_R [\partial Q/\partial x - \partial P/\partial y]\, dA$$

Also $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int\int_R \text{curl}_z\, \mathbf{F}\, dA \quad \text{(curl form of Green's Theorem)}$$

And $$\int_C \mathbf{F} \cdot \mathbf{n}\,ds = \int\int_R \text{div}\, \mathbf{F}\, dA \quad \text{(divergence form of Green's Theorem)}$$

The **Divergence Theorem** (**Gauss's Theorem**) extends the divergence form of Green's Theorem from two to three dimensions. In this case the line integral around a closed curve is replaced by a surface integral around a closed surface and the area integral involving the divergence of **F** is replaced by the volume integral of the divergence of **F** .

2-D

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \text{div}\, \mathbf{F}\, dA \qquad \text{(Divergence form of Green's Theorem)}$$

3-D

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \text{div}\, \mathbf{F}\, dV \qquad \text{(Divergence Theorem)}$$

Also $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$ (Conversion to a surface integral)

Green's theorem gives the relationship between a line integral around a simple closed curve, C, in the x-y plane. **Stokes' Theorem** extends this concept to a closed curve in x-y-z space using the "curl form" of Green's Theorem as follows:

2-D

$$\int_C \mathbf{F}(x,y) \cdot d\mathbf{r} = \iint_R \text{curl}_z\, \mathbf{F}\, dA = \iint_R \text{curl}\, \mathbf{F} \cdot \mathbf{k}\, dA \qquad \text{(curl form of Green's Theorem)}$$

3-D Green's Theorem extended to **Stokes' Theorem** gives :

$$\int_C \mathbf{F}(x,y,z) \cdot d\mathbf{r} = \iint_S \text{curl}\, \mathbf{F} \cdot \mathbf{n}\, dS \qquad \text{(Stokes' Theorem)}$$

where $\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$

Also $\iint_S \text{curl}\, \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \text{curl}\, \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$ (Conversion to a surface integral)