Method of Variation of Parameters

In a Nut Shell: Recall that the method of undetermined coefficients is used to solve for particular solutions of a nonhomogeneous d.e.'s of the form

$$a\, d^2y/dx^2 \;+\; b\, dy/dx \;+\; cy \;=\; f(x)$$

Variation of parameters, gives an alternative way to solve for particular solutions for general types of functions, f(x), beyond those covered using the method of undetermined coefficients.

In general,

Use the Method Undetermined Coefficients for:

Type 1: Second order, linear, ordinary differential equations with constant coefficients, such as:

$$a\, d^2y/dx^2 + b\, dy/dx \;+\; c\, y \;=\; f(x)$$

provided the functions, f(x), are of the type given in the table below.

■ A polynomial	■ Sine functions	■ Cosine functions
■ Sine and Cosine functions	■ Exponential functions	■ Or products of these functions

If f(x) is not among this table of options, then the method of variation of parameters may provide an alternative method to arrive at a particular solution.

Use the Method of Variation of Parameters for:

Type 2A: Second order, linear, ordinary differential equations with constant coefficients where f(x) involves quotients or functions not shown in the above table.

Examples include: $f(x) = \sec x$, $f(x) = \tan x$, $f(x) = 1/x$, $f(x) = x/(x^2 + 1)$, etc.

Type 2B: Second order, linear, ordinary differential equations with variable coefficients such as:

$$x^2\, y''(x) \;+\; 2x\, y'(x) \;-\; y(x) \;=\; f(x) \qquad x > 0$$

Strategy: The method of variation of parameters starts with calculating the complementary solution, y_c, of the d.e. $d^2y/dx^2 + b\,dy/dx + cy = 0$

$$y_c = C_1\,y_1(x) + C_2\,y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are two linearly independent solutions to the homogeneous d.e. In variation of parameters form the particular solution by using two new functions, **u_1(x) and u_2(x)**, yet to be determined, as follows:

$$y_p(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

Next calculate the first derivative of $y_p(x)$ as follows:

$$dy_p(x)/dx = du_1(x)/dx\ \ y_1(x) + du_2(x)/dx\ y_2(x) + u_1(x)\ dy_1(x)/dx + u_2(x)\ dy_2(x)/dx$$

Next comes a key step: To avoid second derivatives of u_1 and u_2 set

$$du_1(x)/dx\ \ y_1(x) + du_2(x)/dx\ \ y_2(x) = 0 \qquad \text{(equation 1)}$$

leaving $dy_p(x)/dx = u_1(x)\ dy_1(x)/dx + u_2(x)\ dy_2(x)/dx$ take another derivative

$$d^2y_p(x)/dx^2 = du_1(x)/dx\ \ dy_1(x)/dx + du_2(x)\ dy_2(x)/dx + u_1(x)\ d^2y_1(x)/dx^2 + u_2(x)\ d^2y_2(x)/dx^2$$

Strategy: Substitute $y_p(x)$, $dy_p(x)/dx$, $d^2y_p(x)/dx^2$ into

$$d^2y/dx^2 + b\,dy/dx + cy = f(x)$$

The result is a second equation to determine $du_{1/}dx$ and du_2/dx which has the form:

$$du_1(x)/dx\ \ dy_1(x)/dx + du_2(x)/dx\ dy_2(x)/dx = f(x) \qquad \text{(equation 2)}$$

where we used that both $y_1(x)$ and $y_2(x)$ satisfy the homogeneous d.e.

$$a\,d^2y/dx^2 + b\,dy/dx + cy = 0$$

The next step is to solve equations (1) and (2), using algebra, for $du_1(x)/dx$ and $du_2(x)/dx$. Note, you have two equations in two unknowns. The unknowns are the $du_1(x)/dx$ and $du_2(x)/dx$.

They equations are as follows: (repeated for you here)

$$du_1(x)/dx \; y_1(x) + du_2(x)/dx \; y_2(x) = 0 \qquad \text{(equation 1)}$$

$$du_1(x)/dx \; dy_1(x)/dx + du_2(x)/dx \; dy_2(x)/dx = f(x) \qquad \text{(equation 2)}$$

Strategy: Integrate expressions for $du_1(x)/dx$ and for $du_2(x)/dx$ to obtain

$u_1(x)$ and $u_2(x)$.

It turns out that the constants of integration end up being absorbed in the complementary solution.

This integration may not be simple.

Finally, the **particular solution** is $y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$

Example: Use the method of variation of parameters to find a particular solution to:

$$y'' + 9y = 2 \sec 3x$$

Strategy: Start with the complementary solution of the homogeneous d.e.

$$y'' + 9y = 0 \qquad (\text{Assume } y(x) = e^{rx})$$

The characteristic equation is: $r^2 + 9 = 0$ and $r = \pm 3i$

So $y_c = C_1 \sin 3x + C_2 \cos 3x$ is the complementary solution of the d.e.

Pick $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$ and recall $y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$

So $y_p(x) = u_1(x) \sin 3x + u_2(x) \cos 3x$

For the particular solution where $u_1(x)$ and $u_2(x)$ are yet to be determined.

Next differentiate $y_p(x)$ which gives

$$y_p' = u_1'(x) \sin 3x + u_2'(x) \cos 3x + 3u_1(x) \cos 3x - 3u_2(x) \sin 3x$$

Key Step: With the method of parameters eliminate the possibility of second derivatives of $u_1(x)$ and $u_2(x)$ by setting the following:

$$u_1'(x) \sin 3x + u_2'(x) \cos 3x = 0 \qquad \text{- - - - -} \quad (1)$$

So the first derivative of y_p reduces to $y_p' = 3u_1(x) \cos 3x - 3u_2(x) \sin 3x$

Next calculate the second derivative of y_p as follows:

$$y_p'' = 3u_1'(x) \cos 3x - 3u_2'(x) \sin 3x - 9u_1(x) \sin 3x - 9u_2(x) \cos 3x$$

and $9\, y_p(x) = 9\, u_1(x) \sin 3x + 9\, u_2(x) \cos 3x$

Substitute y_p'' and $9\, y_p$ into the original d.e. to obtain:

$$2 \sec 3x = 3u_1'(x) \cos 3x - 3u_2'(x) \sin 3x \quad \text{------} \quad (2)$$

Strategy: Use algebra to solve (1) and (2) for $u_1'(x)$ and $u_2'(x)$.

$$u_1'(x) \sin 3x + u_2'(x) \cos 3x = 0 \quad \text{-----} \quad (1)$$

$$3u_1'(x) \cos 3x - 3u_2'(x) \sin 3x = 2 \sec 3x \quad \text{------} \quad (2)$$

The result is:

$$u_1'(x) = 2/3\,, \qquad u_2'(x) = -2/3 \tan 3x$$

Therefore the **particular solution**

$$y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x) \quad \text{is}$$

$$y_p(x) = (2x/3 + B_1) \sin 3x + [\,(2/9) \ln(\cos 3x) + B_2\,] \cos 3x$$

The general solution for y(x) is the sum of the complementary and particular solutions. This is the final result (below).

$$y(x) = C_1 \sin 3x + C_2 \cos 3x + (2x/3) \sin 3x + (2/9) \ln(\cos 3x) \cos 3x$$

(here C_1 and C_2 are constants of integration)

where the constants of integration, B_1 and B_2, are absorbed into C_1 and C_2