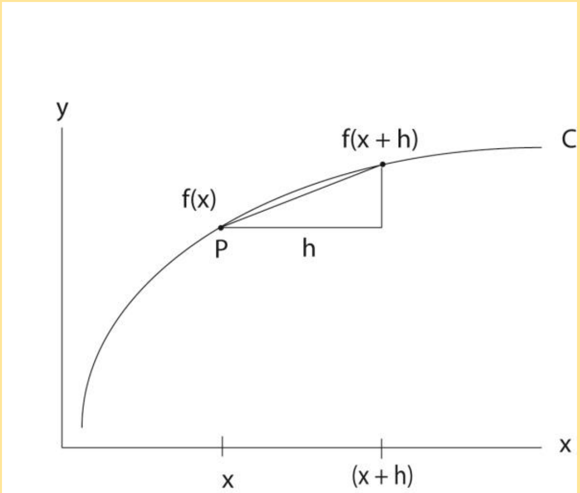
Partial Derivatives of a Function/Gradient of a Function

In a Nut Shell: Recall the derivative of a function of one independent variable (say x) relates directly to the slope of the dependent variable (say y). i.e. dy/dx

In a Nut Shell: Likewise for function z of two independent variables (say x and y), the partial derivative on x gives the slope in the x-direction and the partial derivative on y gives the slope in the y direction. i.e. $\partial z/\partial x$ and $\partial z/\partial y$ This notion of partial derivatives holds for functions of any number of independent variables.

Start by reviewing the definition of the derivative for a function of one independent variable, x

$$df/dx = f'(x) = \lim_{h \to 0} [f(x+h) - f(x)]/h$$



With one independent variable, f '(x) represents the slope of the curve f(x) at point P.

Now consider a function, f(x,y), which has two independent variables x and y.

Strategy: Extend the definition of the limit for a function of one independent variable to a function of two independent variables, x and y, leading to the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ of $f = f(x,y)$.

Definitions of Partial Derivatives of f(x,y) using limits.

The first partial derivative of f(x,y) with respect to x (holding y constant) is:

$$\partial f/\partial x = f_x(x,y) = \lim_{h \to 0} [f(x+h, y) - f(x,y)]/h$$

$$\partial f/\partial x = f_x(x,y) = \lim_{h \to 0} [f(x-h, y) - f(x,y)]/(-h)$$

The first partial derivative of f(x,y) with respect to y (holding x constant) is:

$$\partial f/\partial y = f_y(x,y) = \lim_{h \to 0} [\, f(x, y + h) - f(x,y)] / h$$

$$\partial f/\partial y = f_y(x,y) = \lim_{h \to 0} [\, f(x, y - h) - f(x,y)] / (-h)$$

Second Order Partial Derivatives using Limits

The definitions of second order partial derivatives f_{xx}, f_{yy}, f_{xy}, and f_{yx} of the function,

f(x,y), can be extended using limits as follows:

$$f_{xx} = \partial^2 f(x,y)/\partial x^2 = \lim_{h\to 0} [f_x(x+h, y) - f_x(x,y)] / h$$

$$f_{xx} = \partial^2 f(x,y)/\partial x^2 = \lim_{h\to 0} [f_x(x-h, y) - f_x(x,y)] / (-h)$$

$$f_{yy} = \partial^2 f(x,y)/\partial y^2 = \lim_{h\to 0} [f_y(x,y+h) - f_y(x,y)] / h$$

$$f_{yy} = \partial^2 f(x,y)/\partial y^2 = \lim_{h\to 0} [f_y(x,y-h) - f_y(x,y)] / (-h)$$

$$f_{xy} = \partial^2 f(x,y)/\partial x \partial y = \lim_{h\to 0} [f_x(x,y+h) - f_x(x,y)] / h$$

$$f_{xy} = \partial^2 f(x,y)/\partial x \partial y = \lim_{h\to 0} [f_x(x,y-h) - f_x(x,y)] / (-h)$$

$$f_{yx} = \partial^2 f(x,y)/\partial y \partial x = \lim_{h\to 0} [f_y(x+h,y) - f_y(x,y)] / h$$

Example: Use the table of f(x,y) given below

→ y x ↓	1.8	2.0	2.2
1.5	12.5	10.2	9.3
2.0	18.1	17.5	15.9

and the definitions of the partial derivatives to calculate $f_x(2,2)$, and of $f_{xy}(2,2)$.

Strategy: To calculate the first derivative, apply the definitions of first derivatives using limits

$$f_x = \partial f/\partial x = = \lim_{h \to 0} [f(x + h, y) - f(x,y)] / h \qquad (A)$$

$$f_x = \partial f/\partial x = = \lim_{h \to 0} [f(x - h, y) - f(x,y)] / (-h) \qquad (B)$$

and use the average.

Calculate $f_x (2,2)$: **From table: Using (A)**

$f_x (2,2) = f[(2.5,2) - f(2,2)] / 0.5 = [22.4 – 17.5]/(0.5) = 9.8$

From table: Using (B)

$f_x (2,2) = f[(1.5,2) - f(2,2)] / (-0.5) = [10.2 – 17.5]/ (-0.5) = 14.6$

Calculate the average value for $\partial f/\partial x$ at (2,2). $f_x (2,2) = 12.2$ (result)

Strategy: To calculate the second derivative f_{xy} , apply the definitions of second

derivatives using limits.

$$f_{xy} = \partial^2 f(x,y)/\partial x\partial y = \lim_{h\to 0} [f_x(x,y+h) – f_x(x,y)] / h \qquad (C)$$

$$f_{xy} = \partial^2 f(x,y)/\partial x\partial y = \lim_{h\to 0} [f_x(x,y-h) – f_x(x,y)] /(- h) \qquad (D)$$

Strategy: Apply the definitions of derivatives using limits to find f_{xy} at (2,2).

Need both $f_x(x,y+h)$ and $f_x(x,y-h)$ or $f_x(2,2.2)$ and $f_x(2,1.8)$.

Note: $f_x (2,2)$ previously calculated.

From Table using (A): $f_x (2,1.8) = [20.0 – 18.1]/ (0.5) = 3.8$

From Table using (B): $f_x (2,1.8) = [12.5 – 18.1]/ (-0.5) = 11.2$

So average value of $f_x (2,1.8)$ is $(3.8 + 11.2)/2 = 7.5$

Next calculate $f_x (2,2.2)$ using (A) and (B).

$f_x (2,2.2) = [26.1 – 15.9]/0.5 = 20.4$ and $[9.3 – 15.9]/(-0.5) = 13.2$

The average value of $f_x (2,2.2)$ is $(20.4 + 13.2)/2 = 16.8$. (Result)

Now calculate $f_{xy}(2,2)$ using (C) and (D).

By (C): $f_{xy} = \partial^2 f(x,y)/\partial x\partial y = [f_x(2,2.2) - f_x(2,2)]/h = [16.8 - 12.2]/0.2 = 23.0$

By (D): $f_{xy} = \partial^2 f(x,y)/\partial x\partial y = [f_x(2,1.8) - f_x(2,2)]/h = [7.5 - 12.2]/(-0.2) = 23.5$

So the average value of $f_{xy}(2,2)$ is $(23.0+23.5)/2 = 23.25$ (Result)

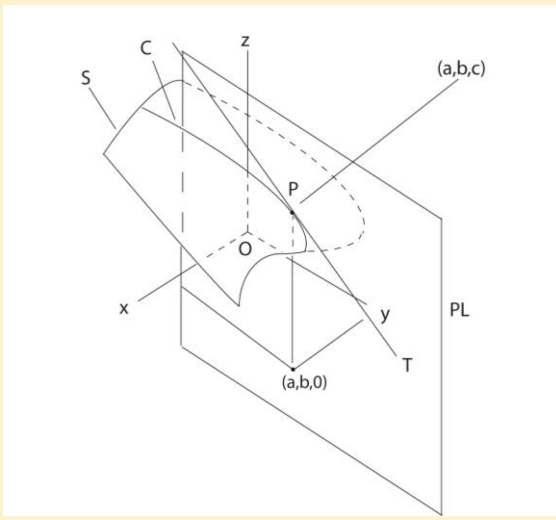
Interpretation as slopes

With a function, $f(x,y)$, of two variables $f_x(x,y)$ represents the slope of $f(x,y)$ in the x-direction whereas $f_y(x,y)$ represents the slope of $f(x,y)$ in the y-direction.

Notation: $f_x(x,y) = \partial f(x,y)/\partial x$ and $f_y(x,y) = \partial f(x,y)/\partial y$

Physical Interpretation of the partial derivative $\partial f(x,y)/\partial y$

Consider a surface S given by $z = f(x,y)$. The intersection of the plane, PL, given by $x = \text{constant}$ with the surface, S, defines the curve of intersection, C. Let the tangent line to C at point P with coordinates (a, b, c) be T.



Then **the slope of the line tangent to the curve C at the point (a,b,c) in the y-direction** is:

$f_y(x,y) = \partial f(x,y)/\partial y$ evaluated at the point P (a,b,c)

Similarly one can imagine a plane, y = constant, intersecting the surface z along a curve D (not shown).

The **slope of the line tangent to the curve D at the point (a,b,c) in the x-direction** is:

$f_x(x,y) = \partial f(x,y)/\partial x$ evaluated at the point P (a,b,c)

Example: For the given function, f(x,y), find $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$, and $f_{yx}(x,y)$

$$f(x,y) = x^2 \exp(-y^2)$$

$\partial f/\partial x = f_x(x,y) = 2x \exp(-y^2)$ (holding y constant)

$\partial f/\partial y = f_y(x,y) = x^2 (-2y) \exp(-y^2)$ (holding x constant)

$\partial/\partial x[\partial f/\partial y] = f_{xy}(x,y) = 2x \exp(-y^2)[-2y] = -4xy \exp(-y^2)$

$\partial/\partial y[\partial f/\partial x] = f_{yx}(x,y) = (2x)[-2y \exp(-y^2)] = -4xy \exp(-y^2)$

For continuous functions $f_{xy}(x,y) = f_{yx}(x,y)$ **The order of differentiation is irrelevant.**

Example: The concept of partial derivatives can be extended to a function of more than two independent variables as shown in this example.

Given: $f(x, y, z) = \cos(4x + 3y + 2z)$ Find: $\partial^3 f/\partial x\partial y\partial z = f_{xyz}$

Start with $\partial f/\partial x = -4 \sin(4x + 3y + 2z)$

Then take the next derivative with respect to y giving $\partial^2 f/\partial x\partial y = -12 \cos(4x + 3y + 2z)$

Finally take the derivative with respect to z gives $\partial^3 f/\partial x\partial y\partial z$

$$\partial^3 f/\partial x\partial y\partial z = 24 \sin(4x + 3y + 2z) \quad \text{(result)}$$

Note that the cosine function is continuous. So the order of differentiation should be immaterial.

Let us check this out by calculating $\partial^3 f/\partial z\partial y\partial x$

Start with $\partial f/\partial z = -2 \sin(4x + 3y + 2z)$

Then $\partial^2 f/\partial z\partial y = -6 \cos(4x + 3y + 2z)$

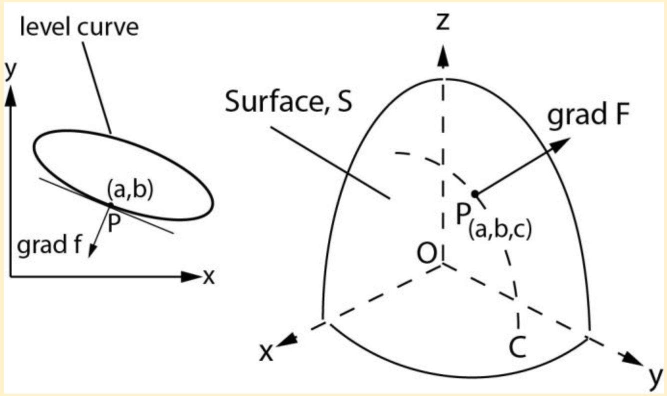
And finally $\partial^3 f/\partial x\partial y\partial z = 24 \sin(4x + 3y + 2z)$ (same result)

Gradient of a Function

In a Nut Shell: Suppose a level curve is defined by f(x,y) = 0. Then the gradient of f(x,y) is defined as the vector grad f(x,y) = $\partial f/\partial x$ **i** + $\partial f/\partial x$ **j** . Grad f points in the direction that gives the greatest change of f.

Suppose a surface is defined as F(x,y,z) = 0. Then the gradient of F(x,y,z) is defined as the vector grad F(x,y,z) = $\partial f/\partial x$ **i** + $\partial f/\partial x$ **j** + $\partial f/\partial z$ **k**. Grad F points in the direction that gives the greatest change of F.

The figures shown below illustrate these vectors.



Note that grad f is normal to the level curve, in this case at the point (a,b). (Figure on the left)
One application might be to find the tangent line to the level curve at a specified point.

Likewise, note that grad F is normal to the surface at point (a,b,c). (Figure on the right)
One application might be to find the tangent plane to the surface at a specified point.

Example: Find the tangent plane to the surface $z = \sin(2x + 2y)$ at the point $(\pi/2,\pi/2,0)$.

$$F(x,y,z) = \sin(2x+2y) - z = 0$$

$$\text{Grad } F = 2\cos(2x + 2y)\,\mathbf{i} + 2\cos(2x + 2y)\,\mathbf{j} - \mathbf{k}$$

So the normal to the plane at the point is $\mathbf{n} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

The general equation for a plane is $(\mathbf{r} - \mathbf{r}_o) \cdot \mathbf{n} = 0$

Here $\mathbf{r} = \langle x,y,z \rangle$ and $\mathbf{r}_o = \langle \pi/2,\pi/2,0 \rangle$

So $[\,(x - \pi/2)\,\mathbf{i} + (y - \pi/2)\,\mathbf{j} + z\,\mathbf{k}\,] \cdot [\,2\mathbf{i} + 2\mathbf{j} - \mathbf{k}\,] = 0$

$2x + 2y - z - 2\pi = 0$ (result)

Example: Find the maximum rate of change of the function, f(x,y,z) at the specified point.
Also find its direction. $f(x,y,z) = \tan(x + y + 2z)$ at the point (-1,-1,1)

grad $f = \langle \sec^2(x+y+2z),\ \sec^2(x+y+2z),\ 2\sec^2(x+y+2z) \rangle$

At the specified point grad $f = \langle \sec^2(0), \sec^2(0), 2\sec^2(0) \rangle = \langle 1, 1, 2 \rangle$

So the maximum rate of change of the function = $\sqrt{6}$. (result)

And its direction at the point (-1,-1,1) is $\langle 1, 1, 2 \rangle$. (result)