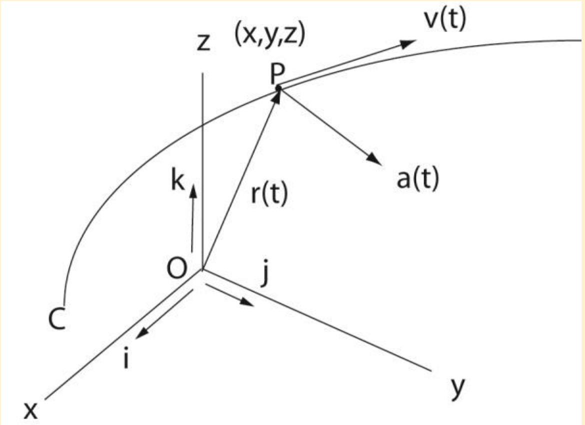
Motion of a Particle in Space

In a Nut Shell: A particle can be located by its position vector, **r**, in space. Description of its motion involves both its velocity vector, **v**, and its acceleration vector, **a**

Strategy: Let **r** = x **i** + y **j** + z **k** be a position vector from the origin, O, to an arbitrary point P(x,y,z) (particle) on a curve, C, in space . Then d**r** / dt is a vector tangent to this curve. This curve, C, represents the path of motion of the particle, P, in space.



Take the derivative of the position vector, **r** , to obtain the velocity of the particle, **v** .

v = d**r** / dt = **velocity of the particle** along its path

So **v** = d**r** / dt = dx/dt **i** + dy/dt **j** + dz/dt **k**

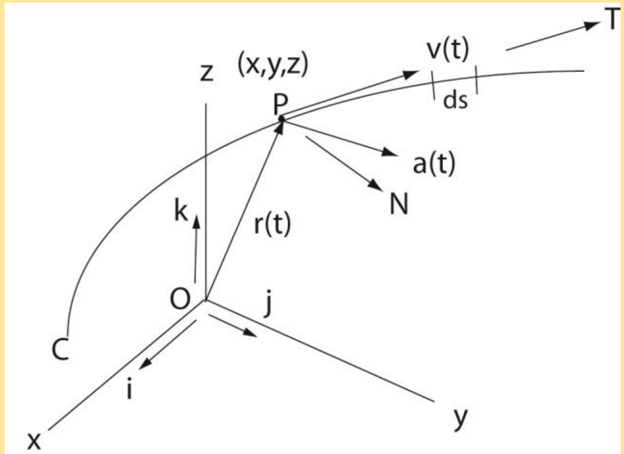
where dx/dt, dy/dt, and dz/dt represent the x, y, and z-components of velocity of the particle moving along C.

Strategy: Take the derivative of the velocity of the particle to obtain its acceleration.

So **a** = d**v** / dt . In “rectangular coordinates” x, y, z

$$\mathbf{a} = d\mathbf{v}/dt = d^2x/dt^2\,\mathbf{i} + d^2y/dt^2\,\mathbf{j} + d^2z/dt^2\,\mathbf{k}$$

In a Nut Shell: It is often convenient to describe motion of a particle in terms of its normal and tangential components. Let **T** be the unit tangential vector to the path, C, of motion of a particle and let **N** be its unit normal vector.



Then since **v** , the velocity of the particle is always tangent to its path

$$\mathbf{v} = v\,\mathbf{T} \quad \text{so} \quad \mathbf{a} = d\mathbf{v}/dt = dv/dt\ \mathbf{T} + v\ d\mathbf{T}/dt$$

here dv/dt represents **the tangential component of acceleration of the particle**

Now $d\mathbf{T}/dt = [d\mathbf{T}/ds]\,[ds/dt]$ using the chain rule of differentiation. So

$$v\ d\mathbf{T}/dt = v\,d\mathbf{T}/ds\ v = v^2\ d\mathbf{T}/ds \qquad \text{since} \quad ds/dt = v$$

Note: Since $\mathbf{T} \cdot \mathbf{T} = 1$ (**T** is a unit tangent vector to the path)

$\mathbf{T} \cdot d\mathbf{T}/ds = 0$ which shows that $d\mathbf{T}/ds$ is perpendicular to **T** .

Define the curvature of C as $|\,d\mathbf{T}/ds\,| = \kappa$ and let **N** be the unit normal vector to C

$d\mathbf{T}/ds = \kappa\ \mathbf{N}$ so the acceleration of the particle becomes (where κ = curvature)

$\mathbf{a} = d\mathbf{v}/dt = dv/dt\ \mathbf{T} + \kappa v^2\ \mathbf{N}$, $\kappa = 1/\rho$, ρ = radius of curvature

Note: The acceleration in general has components tangential and normal to its path.

Note further: To find: $\mathbf{T} = \mathbf{v}/|\mathbf{v}|$ $\qquad \mathbf{a}\cdot\mathbf{a} = a^2 = a_T{}^2 + a_N{}^2$

To find: **N** $\qquad \mathbf{N} = [d\mathbf{T}/ds]/\kappa = [d\mathbf{T}/ds]/|\,d\mathbf{T}/ds\,| = [d\mathbf{T}/dt]/|\,d\mathbf{T}/dt\,|$

Example: Given the position vector, $\mathbf{r} = t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$ where **i , j, k** are unit vectors (base vectors) along the x, y, and z axes and t is any arbitrary time

find: the velocity of the particle, **v,** the unit tangential vector, **T** , the unit normal vector, **N**, the curvature of the particle's path, κ **,** and the acceleration, **a**, of the particle.

Strategy: The derivative of the position vector, $\mathbf{r}$, gives the velocity of the particle.

So $\mathbf{v} = d\mathbf{r} / dt = \mathbf{i} + \cos t\, \mathbf{j} - \sin t\, \mathbf{k}$ (result)

Since the velocity is always tangent to its path, the unit tangential vector can be found by dividing the velocity vector by its magnitude.

$$\mathbf{T} = \mathbf{v} / |\mathbf{v}| = [\mathbf{i} + \cos t\, \mathbf{j} - \sin t\, \mathbf{k}] / \sqrt{(1^2 + \cos^2 t + \sin^2 t)}$$

Therefore $|\mathbf{v}| = v = \sqrt{2}$

So $\mathbf{T} = [\mathbf{i} + \cos t\, \mathbf{j} - \sin t\, \mathbf{k}] / \sqrt{2}$ (result)

Recall that the rate of change of the unit tangential vector along its path, $d\mathbf{T}/ds$

is given by the product of the curvature of the path, κ, with its unit normal vector, $\mathbf{N}$.

So $d\mathbf{T}/ds = \kappa\, \mathbf{N}$

and $|d\mathbf{T}/ds| = \kappa$ since $|\mathbf{N}| = 1$

Next find the unit normal vector, $\mathbf{N}$; recall $\mathbf{N} = [d\mathbf{T}/ds] / \kappa$

But $\kappa = |d\mathbf{T}/ds|$

Therefore $\mathbf{N} = d\mathbf{T}/ds \,/\, |d\mathbf{T}/ds| = [d\mathbf{T}/dt] / |d\mathbf{T}/dt|$

and

$d\mathbf{T}/dt = [-\sin t\, \mathbf{j} - \cos t\, \mathbf{k}] / \sqrt{2}$ and $|d\mathbf{T}/dt| = 1 / \sqrt{2}$

So $[d\mathbf{T}/dt] / |d\mathbf{T}/dt| = [-\sin t\, \mathbf{j} - \cos t\, \mathbf{k}] = \mathbf{N}$ (result)

Check: The unit tangential and normal vectors are perpendicular. Therefore the dot product should be zero.

Is $\mathbf{T} \cdot \mathbf{N} = 0$?

$$\{[\mathbf{i} + \cos t\, \mathbf{j} - \sin t\, \mathbf{k}] / \sqrt{2}\} \cdot \{[-\sin t\, \mathbf{j} - \cos t\, \mathbf{k}]\}$$

$= -\cos t \sin t + \sin t \cos t = 0$ (check)

Next find the curvature, κ .

$\kappa = |d\mathbf{T}/ds| = |d\mathbf{T}/dt|\, dt/ds$ using the chain rule

Now $dt/ds = 1/v$

$\kappa = |d\mathbf{T}/ds| = |d\mathbf{T}/dt|\, (1 / v) = (1 / \sqrt{2})(1 / \sqrt{2}) = ½$ (result

Finally, the acceleration of the particle is the rate of change of the velocity of the particle, $d\mathbf{v}/dt$. Recall $\mathbf{v} = \mathbf{i} + \cos t\,\mathbf{j} - \sin t\,\mathbf{k}$. So

$\mathbf{a} = -\sin t\,\mathbf{j} - \cos t\,\mathbf{k}$ (result in Cartesian components)

Next calculate the normal and tangential components of acceleration.

Strategy: Use the dot product to find the tangential component of acceleration.

$$\mathbf{a} \cdot \mathbf{T} = a_T$$

where $\mathbf{a} = -\sin t\,\mathbf{j} - \cos t\,\mathbf{k}$ and $\mathbf{T} = [\,\mathbf{i} + \cos t\,\mathbf{j} - \sin t\,\mathbf{k}\,] / \sqrt{2}$

So $a_T = (-\sin t)(\cos t) + (-\cos t)(-\sin t) = 0$ (result)

Finally calculate the normal component of acceleration of the particle.

Recall $\mathbf{a} = a_N\,\mathbf{N} + a_T\,\mathbf{T}$

So $|\mathbf{a}| = \sqrt{(a_N)^2 + (a_T)^2} = \mathbf{a} \cdot \mathbf{a}$

$$a_N = \sqrt{\mathbf{a} \cdot \mathbf{a} - (a_T)^2}$$

In this example $a_T = 0$ so $a_N = (-\sin t\,\mathbf{j} - \cos t\,\mathbf{k}) \cdot (-\sin t\,\mathbf{j} - \cos t\,\mathbf{k})$

Therefore $a_N = \sin^2 t + \cos^2 t = 1$ (result)

Summary

$$\mathbf{r} = t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$$

$$\mathbf{v} = \mathbf{i} + \cos t\,\mathbf{j} - \sin t\,\mathbf{k} = \sqrt{2}\,\mathbf{T}$$

where $\mathbf{T} = (\,\mathbf{i} + \cos t\,\mathbf{j} - \sin t\,\mathbf{k}\,) / \sqrt{2}$

$$\mathbf{a} = -\sin t\,\mathbf{j} - \cos t\,\mathbf{k}$$

$\mathbf{a} = (1)\,\mathbf{N}$ where $\mathbf{N} = [\,-\sin t\,\mathbf{j} - \cos t\,\mathbf{k}\,]$